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Nonholonomic manifolds and nilpotent analysis

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To I.M. Gel'fand for his 75 anniversary with admiration

Abstract. *This paper is dedicated to the exposition of the theory of nonholonomic manifolds. This exposition includes the following topics:*
– geometry and classification of distributions
– algebraic structures (nilpotent Lie algebras and nilpotentization)
– nonholonomic dynamic systems and nonholonomic Riemann geometry
– hypoelliptic operators.
The systematic exposition of the nilpotent analysis apparatus allow to review all the old and resent results from the unique point of view.

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Key-Words: *Distribution on manifolds, nonholonomic geometry, Nilpotentization, Carnot-Caratheodory metric, nonholonomic geodesic flow, centaur, Lagrangian-Hamiltonian formalism.*
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INTRODUCTION

We shall use the term "nilpotent analysis" as a synonym of the longer expression "analysis on smooth manifolds with completely nonintegrable regular distributions", i.e. analysis on nonholonomic manifolds (the origin of the word "nilpotent" in the present context will be explained later). This subject includes geometry of these distributions, the basic notions of analysis on nonholonomic manifolds (differentials, Hessians, singular points), dynamic systems, generated by nonholonomic geodesic flows, metric problems, nonholonomic Riemann geometry and the theory of differential and pseudodifferential operators (spectral problems for hypoelliptic operators, hypoelliptic diffusion, hypohyperbolic operators etc.). The collection of these topics constitutes the subject of the present paper. This survey (as well as numerous other papers [1-15]) shows that this topic is closely connected with singularity theory, random walks, control theory, Riemannian geometry, partial differential equations theory and some other regions of mathematics.

Do not include historical questions in the present survey, therefore we do not mention earlier works, referring the interested reader to the historical survey in [17]. Several problems from natural science must be also mentioned among the sources of the present theory, such as mechanical and optical nonholonomic problems, optimal control, quantization of systems with constraints.

The role of nilpotent algebras and Lie groups in this theory has been repeatedly mentioned; however, this role was revealed only in the simplest cases. We ascribe categorial sense to these structures. Completely nonholonomic distribution induces an equivalent structure of homogeneous nilpotent Lie algebras on the band of the tangent bundle (nilpotentization). It turns out that systematic use of this structure allows to expose (in a uniform manner) a lot of results obtained earlier approximation theorem for nonholonomic ϵ-ball, principal term theorem for Weyl formulas for hypoelliptic operators etc).

Consequent use of the bundle of nilpotent Lie algebras leads to a revision of main notions of analysis, starting with such notions as differential, critical point etc. Note that the analogy with superanalysis, that seems natural at first sight, is not completely precise, because nilpotent analysis is based on a different generalization of standard commutative analysis superanalysis revises the notion of a variable, while nilpotent analysis revises the notion of a jet (i.e. derivatives). The simplest and most investigated case of nonholonomic manifolds is the case of contact structures, where Heisenberg algebra plays the role of the nilpotent algebra. However, even for this case metrical and variational problems are poorly investigated.

The present paper does not pretend to be a complete exposition of nonholonomic analysis, we consider only as a convenient instrument for solving problems from nonholonomic geometry. The contents of this paper can be seen from the above titles of points. The first part is devoted to the consequent exposition of the main notions of analysis on nonholonomic manifolds: geometry of distributions (1), construction of the bundle of nonholonomic Lie algebras on nonholonomic manifolds (2); necessary information about the homogeneous nilpotent Lie algebras and their classification (3). In 5 and 6 we describe a graded Lie algebra of vector fields jets on a nonholonomic manifold; before that we prepare a formal model, necessary for that description (4). In 7-9 we explain the questions, connected with nonholonomic Riemann metric; in 7 its definition is given, in 8 asymptotics of ϵ-balls is described, in 9 formula for the Hausdorff dimension of a nonholonomic manifold is given. The last (10) of this part is devoted to the exposition of basic notions of nilpotent analysis (first and second differentials of a function on a nonholonomic manifold, critical point, nonholonomic Hessian).

The beginning of a second part (11-15) is devoted to nonholonomic Riemann geometry and to dynamical systems, generated by a nonholonomic geodesic flow. In 11 a Hamilton formalism is presented, that allows to define a flow in the contangent bundle, and a Lagrange formalism, defining a flow in the "Centaurus" – a mixed distribution, that is a direct sum of the original distribution and the codistribution, that is its annihilator in the cotangent bundle. Nonholonomic geodesic equations are given (12). Reduction of nonholonomic geodesic ($\mathcal{N}g$) flow of Lie groups is described, and complete description of

$\mathcal{N}$g-flows is given for the simplest case – the flow on 3-dimensional Heisenberg group (13). The ergodic theorem for a $\mathcal{N}$g-flow on $SL_2\mathbb{R}$ is given (14), and the description of singularities of non-holonomic wave fronts for 3-dimensional Lie groups is given in 15.

The remaining paragraphs of this part are devoted to the questions, connected with nonholonomic Laplacian: its definition is given in 16, a theorem on hypoharmonic functions – in 17, nonholonomic Green formula – in 18. The last two paragraphs (19-21) are connected with formulating a conjecture about the main term in Weyl formula for nonholonomic Laplacian: in 19 the main quasi-homogeneous part of a nonholonomic Laplacian is constructed, and Metivier theorem about asymptotics of spectral function is given; in 20 the connection between the asymptotic behaviour of eigenvalues' growth and non-holonomic diffusion is given.

The present survey is summarizing the earlier papers by the authors [16-25] but the present exposition is independent on that works and it contains new results. It contains also a systematic exposition of a new apparatus which allows to review both the new and recent results from the unique point of view. The authors are new working further papers on nilpotent analysis and problems of nonholonomic geometry. Note that our reference list is in no sense complete, because it cannot be made complete without abnormal increase of its volume; references in the text are also sometimes not complete.

The authors are happy to dedicate this paper to 75-th anniversary of Izrail Moiseevich Gel'fand, a remarkable scientist, thinker, teacher of mathematicians. whose personality and whose talent attract attention and cause great interest of mathematicians of different countries, specializations and ages.

Part I Nilpotent analysis

1. DISTRIBUTIONS' GROWTH VECTOR

By a *distribution* V on a smooth manifold M one understands a smooth subbundle $V = \{V(x), x \in M\}$ of a tangent bundle TM. A vector field ξ on M is called *admissible* with respect to the distribution V if $\xi(x) \in V(x)$ for arbitrary point $x \in M$.

For every point $x \in M$ construct a chain $V(x) = V_1(x) \subset V_2(x) \subset \dots$ of linear spaces in a tangent space T_xM defining $V_i(x)$ as a linear envelope of all the values (in this point x) of vector fields, that can be represented by Lie brackets of length $\leqslant i$, of admissible vector fields. So $V_2 = [V_1, V_1], \dots, V_i = [V_{i-1}, V_1]$. By a *growth vector* of a distribution V in the point x we denote a sequence of integers $\{n_i(x)\}$, where $n_i(x) = \dim V_i(x)$. Will call the distribution *regular*, if for every i the function $n_i(x)$ is constant on the whole manifold M. For regular distributions the set of linear spaces $\{V_i(x), x \in M\}$ forms a distribution V_i on M. This chain of distributions $V_1 \subset V_2 \subset \ldots$ will be called a *Lie flag* of a regular distribution. (In the present paper by a distribution we'll understand regular distributions – except several cases, when the opposite will be explicitely stated. In order to investigate nonregular distributions one needs a new notion of a differential system, more general than the one used in the present paper, see [17]).

We'll call a distribution completely nonholonomic, if, starting from some i_0, the equality $V_i = TM$ is true. The smallest such i_0 is called the nonholonomity degree of the distribution V; usually will denote this degree by $k = k^V$.

On the set of all growth vectors a natural partial ordering can be defined: $\{n_i^V\} > \{n_i^W\}$, if for all i inequalities $n_i^V \geqslant n_i^W$ are true. Therefore we can speak about the maximal element in the set of all growth vectors, corresponding to distributions of given dimension in $\mathbb{R}^n$.

The notion of regularity, complete nonholonomity etc, already defined for distributions, can be naturally transferred to germs and jets of distributions. In particular, germs of regular distributions admit the following characterization.

PROPOSITION 1.1. A germ (jet) of a distribution V in a point $x \in M$ is regular iff the module, formed by germs (jets) of V-admissible vector fields, is free over the ring of germs (respectively jets) of smooth functions on M.

Growth vector is one of the main characteristics of distribution. Let's first describe germs of distributions of maximal growth. By S_n^m we denote the set of all germs of m-dimensional distributions in the point $O \in \mathbb{R}^n$.

THEOREM 1.2. 1. All distributions of maximal growth from S_n^m have one and the same growth vector (hence one and the same nonholonomity degree).

2. The components $\{\tilde{n}_i\}$ of the maximal growth vector from S_n^m up to the k = nonholonomity degree coincide with the dimensions growth of homogeneous component in free Lie algebra with m generators, i.e. the following formula is true:

$$\tilde{n}_i = \dim \mathfrak{g}_i$$

where $\mathfrak{g}_i = \bigoplus_{j \leqslant i} \mathfrak{g}^j$ and $\mathfrak{g}^j$ is an j-th homogeneous component of the free Lie algebra $\mathfrak{g}$ with m generators.

Let's denote nonholonomity degree of a distribution of maximal growth from S_n^m by $k_{n,m}$. Due to theorem 1.2 one can calculate $k_{n,m}$ using a well-known formula for the dimensions of homogeneous components of free Lie algebra with $m \cdot n$, generators (see, e.g. [26]):

$$(1.1) \qquad \tilde{n}_j - \tilde{n}_{j-1} = \frac{1}{j} \sum_{d/j} \mu(d)\; n_1^{j/d}$$

This formula shows that for fixed n_1 the dimension $\tilde{n}_j$ of j-th component has the following asymptotics ($n_1 = m$):

$$\tilde{n}_j \sim \frac{n_1^j}{j}$$

and therefore for fixed m we obtain the following asymptotics describing how nonholonomity degree grows with n:

$$k_{n,m} \underset{n\to\infty}{\sim} \log_m n.$$

One can identify the set S_n^m with the set of germs of sections of the trivial bundle

$$\delta : U \to U \times Gr_n^m$$

where U is a germ in $O \in \mathbb{R}^n$, Gr_n^m is a Grassman manifold of m-dimensional planes in $\mathbb{R}^n$. This identifications allows to define a structure of infinite-dimensional manifold on S_n^m namely a manifold on which Whitney C^∞-topology is defined (for its definition see e.g. [27]). Typical distribution germs are described as follows.

THEOREM 1.3. Distributions of maximal growth form an open everywhere dense subset of S_n^m in the sense of Whitney's C^∞-topology.

COROLLARY. A germ of a distribution $V \in S_n^m$ in $O \in \mathbb{R}^n$ that has maximal growth in O, is regular and completely nonholonomic.

For degenerate distributions two groups of problems arize. The first one is connected with distributions, whose growth is less than maximal in isolated point, while in the neighbourhood of this point the distribution has a maximal growth. These problems include description of degenerations of small codimensions of their stratifications, possibility of some normal form for jets of such distributions etc. Some of these problems are analysed in [24].

The second group consists of problems, connected with regular distributions of non-maximal growth. Codimensionality of the set of such distributions in S_n^m is ∞. In this case adequate calculations of dimensions and codimensions must be organized in terms of functional modules – their total number, the number of variables in them. Due to page limit we have no possibility to deal with these problems in the present survey, therefore we refer the interested reader to the paper [21].

We also want to point out a special and very important branch of the theory of regular distributions of nonmaximal growth namely, investigating left-invariant distributions on Lie groups (or their homogeneous spaces). In this case the distribution $V = \{V(x), x \in G\}$ is uniquely determined by a linear subspace of a Lie algebra $\mathfrak{g} = T_e G$ and so the set of such distributions is finite - dimensional. Here such problems arise as describing maximal growth for various types of Lie groups, and describing stratification of degenerations. The authors do not know any publications devoted to these problems.

2. NILPOTENTIZATION (*)

By a *nonholonomic* (smooth) *manifold* we'll understand a pair (M, V), where M is a smooth manifold, V – a regular and completely nonholonomic distribution on M (cf. 1). Nonholonomic manifolds form a category (we'll denote it by $\mathcal{NM}$) where morphisms $(M_1, V_1) \to (M_2, V_2)$ are smooth mappings of manifolds $M_1 \overset{\varphi}{\to} M_2$ transforming one distribution into another. Every smooth manifold M can be consider as nonholonomic with $V = TM$, so the category of smooth manifolds can be considered as a full subcategory of the category $\mathcal{NM}$. Another important subcategory of $\mathcal{NM}$ is a category $\mathcal{N}G$ of nonholonomic Lie groupe, whose objects are pairs (G, V), where G is a Lie group, and V-left-invariant completely nonholonomic distribution on G. Local versions of categories $\mathcal{NM}$ and $\mathcal{N}G$ can be naturally defined: namely, the categories of germs of nonyolonomic smooth manifolds and of nonholonomic local Lie groups. Due to classical Lie theorem (see, e.g. [26]), that establishes correspondence between local nonholonomic Lie groups and nonholonomic Lie algebras, the objects of this category are pairs $(\mathfrak{g}, \mathfrak{v})$, where $\mathfrak{g}$ is a Lie algebra, $\mathfrak{v}$ - a linear subspace of $\mathfrak{g}$, generating $\mathfrak{v}$ as a Lie algebra.

Our goal is to prove the following theorem.

THEOREM. The regular distribution on a smooth manifold M define the canonical bundle of nilpotent homogeneous Lie algebras $\mathfrak{g}_V M = \{\mathfrak{g}_x\}$ over M.

Remark. This bundle can be identified noncanonically with tangent bundle TM (see 7). The identification is any point: $T_x M \approx \mathfrak{g}_x$ is unique up to an automorphism of nonholonomic Lie algebras $\mathfrak{g}_x$.

The theorem is based on the following construction. Let us introduce the vecto space

$$\mathfrak{g}_x = \Sigma\ V_i(x)/V_{i-1}(x).$$

(*) See [42].

LEMMA. The following definition of Lie brackets is correct. Denote

$$\hat{V}_i = V_i(x)/V_{i-1}(x),$$

$\xi \in V_i$ we'll denote the image of ξ in $\hat{V}_i$ by $\hat{\xi}_i$. For arbitrary $\hat{\xi} \in \hat{V}_i$ and $\hat{\psi} \in \hat{V}_j$ we define a Lie bracket by a formula

$$[\hat{\xi}, \hat{\psi}] = \widehat{[\xi, \psi]} \in \hat{V}_{i+j}$$

where ξ, ψ are representatives of $\hat{\xi}$ and $\hat{\psi}$ in V_i and V_j correspondently.

We'll say that the ideal of relations in a graded Lie algebra is *homogeneous,* if together with arbitrary element it contains all its homogeneous components.

Nilpotent Lie algebras $\mathfrak{g}_x$ form a bundle over M; we'll denote it by $\mathfrak{g}_V M$. We'll say that a nilpotent Lie *algebra* is *homogeneous* if it is a factoralgebra of a free algebra over the homogeneous ideal.

Summarizing all that we obtain the following statement.

PROPOSITION 2.1. Tangent bundle of a nonholonomic manifold (M, V) has a canonic structure of a bundle of homogeneous nilpotent Lie algebras $\mathfrak{g}_V M$ so that morphism of nonholonomic manifolds $\varphi : (M, V) \to (\tilde{M}, \tilde{V})$ corresponds to morphism of bundles $\mathfrak{g}_V M \overset{d\varphi}{\to} \mathfrak{g}_V \tilde{M}$. The bundle of Lie algebras $\mathfrak{g}_V M$ over a nonholonomic manifold (M, V) allows to construct a bundle of nonholonomic Lie groups $G_V M$, whose bands are Lie groups G_x, corresponding to the algebra $\mathfrak{g}_x$. We'll say that this group G is an *osculating Lie group* of a nonholonomic manifold (M, V) in a point x.

Bundle morphism $d\varphi : \mathfrak{g}_V M \to \mathfrak{g}_V \tilde{M}$ will be called a differential of a mapping φ of nonholonomic manifolds. The differential $d\varphi$ is a homomorphism of Lie algebras $\mathfrak{g}_x \to \mathfrak{g}_{\varphi(x)}$ on every band, a homomorphism, smoothly depending on a point x of a manifold.

Note that over for different points x of the manifold corresponding Lie algebras $\mathfrak{g}_x$ have the same dimensions vector (formed by dimensions of homogeneous components), but, generally speaking, they are not isomorphic, in other words, $\mathfrak{g}_V M$ is not a locally trivial bundle of Lie algegras over M. In view of this fact the notion of smooth dependence on the point $x \in M$ needs some comment.

Let U be a sufficiently small neighbourhood of a point x Let's fix a trivialization $TU = U \times \mathbb{R}^n$. Let's consider all possible nilpotent homogeneous Lie algebras, for which the underlying linear space coincides with $\mathbb{R}^n$, and growth vector – with N. Denote the set of all such algebras by Nil_N.

PROPOSITION 2.2. Germs of nonholonomic manifolds with given growth vector have a structure of infinite-dimensional smooth manifold, that can be naturally identified with the manifold of germs of smooth sections $U \to U \times Nil_N$.

This identification allows to speak about smooth dependence of the above-constructed Lie algebras $\mathfrak{g}_x$ and their homomorphisms etc. on the point $x \in M$. In (3) the set Nil_N will be analysed in more detail, and in the present point we restrict ourselves to the description of a bundle $Nil_N \to Fl_N$ where Fl_N is a space of flags with growth vector N.

Every nilpotent homogeneous Lie algebra $\mathfrak{g} \in Nil_N$ defines on $\mathbb{R}^n$ a flag of subspaces $W_1 \subset W_2 \subset \ldots$

dim $W_i = n_i$ whose members are elements of natural filtration on $\mathbb{R}^n$. Among all such flags $\{w_i\} \in Fl_N$ a canonic flag $fl_N = \{\mathbb{R}^{n_1} \subset \mathbb{R}^{n_2} \subset \ldots\}$ is naturally selected. By $\tilde{N}il_N$ will denote the set of all elements of Nil_N with flag fl_N.

PROPOSITION 2.3. Assume $N = \{n_i\}_{i=1}^k$ is a realizable growth vector. Then the following statements are true:

1. The manifold Nil_N is a direct product

$$Nil_N = \tilde{N}il_N \times Fl_N$$

2. Fl_N is a homogeneous space of the group GL_n; namely $Fl_N = GL_n / \mathcal{P}_N$ where $\mathcal{P}_N$ is a subgroup of blockwise uppertriangular matrices with blocks of sizes $\{n_i - n_{i-1}\}_{i=1}^k$.

3. $\tilde{N}il_N$ is an algebraic manifold of structural constants.

3. HOMOGENEOUS NILPOTENT LIE ALGEBRAS

Assume $\mathfrak{g}$ is a homogeneous nilpotent Lie algebra with growth vector $N = \{n_i\}$. The complement to $[\mathfrak{g}, \mathfrak{g}]$ in $\mathfrak{g}$ is called a *standard linear space,* and the correspondent left-invariant distribution on Lie group $\mathfrak{g}$ is called a *standard distribution.*

PROPOSITION 3.1. 1) Automorphisms group of Lie algebra: $\mathfrak{g}$ acts transitively on the set of all standard subspaces of $\mathfrak{g}$

2) For free k-step for every k the following exact sequence

$$0 \to [\mathfrak{g}, \mathfrak{g}] \to \mathfrak{g} \to \mathfrak{g} /_{[\mathfrak{g}\,\mathfrak{g}]} \to 0$$

determines the factorization of Aut $\mathfrak{g}$ into the semidirect product

$$\text{Aut}\ \mathfrak{g} = GL_{n_1} \ltimes \mathbb{R}^{n_1 \times (n - n_1)}.$$

The elements of the type $(g; 0)$ correspond to automorphisms of standard space.

Further on by a nilpotent nonholonomic Lie group we'll always understand a nilpotent Lie group with a standard distribution (of course, only in case some other distribution is not explicitly defined).

A nilpotent homogeneous Lie algebra, considered as a linear space, can be decomposed into a direct sum of homogeneous components

$$\mathfrak{g} = \mathfrak{g}^1 \oplus \mathfrak{g}^2 \oplus \mathfrak{g}^3 \oplus \ldots \oplus \mathfrak{g}^k$$

where $\mathfrak{g}^{i+j} = [\mathfrak{g}^i, \mathfrak{g}^j]$. This decomposition allows to define on $\mathfrak{g}$ a one-parametric automorphisms group $\{h_t\}_{t \in \mathbb{R}^*}$, that is an image of multiplicative group $\mathbb{R}^*$ for the diagonal imbedding

$$\alpha \mapsto \begin{pmatrix} \alpha & & 0 \\ & \ddots & \\ 0 & & \alpha \end{pmatrix} \in \mathrm{Aut}\, \mathfrak{g}_x$$

Transformations h_t are called dilatations. Automorphism h_t coincides with multiplication on t^j on each component $\mathfrak{g}^j$, and can be extended to the whole $\mathfrak{g}$ by linearity.

For the following it is important to single out cases when the bundle $\mathfrak{g}_V M$ is locally trivial.

a) Distributions of general position

Assume the germ of a distribution of dimensional n_1 in M is of maximal growth we'll call the *dimension* M exact, if it coincides with $\tilde{n}_k = \dim \tilde{\mathfrak{g}}_k$ for some k, where $\tilde{\mathfrak{g}}$ is a free Lie algebra with n_1 generators (cf. 1).

If the growth V is maximal and the dimension n is exact, then for every x the algebra $\mathfrak{g}_x$ is free nilpotent Lie algebra of k-th step with n_1 generators, and the bundle $\mathfrak{g}_V M$ is locally trivial.

In case the growth V is maximal, but dimension of M is not exact, Lie algebras $\mathfrak{g}_x$ are, so to say, "free up to the last step", that is, on this last step nontrivial relations arize, that depend on the point $x \in M$. To be more precise, let's denote by Θ_k^{k-1} a functor, that transforms a homogeneous nilpotent k-step Lie algebra $\mathfrak{g}$ into $(k-1)$-step Lie algebra that is obtained from $\mathfrak{g}$ by a factorization modulo the commutative ideal $\mathfrak{g}^k$. This functor Θ_k^{k-1} induces the functor on the bundle of nilpotent algebras on the manifold. These considerations mean that bundle $\Theta_k^{k-1}\ \mathfrak{g}$ is locally trivial. The germ of a bundle $\mathfrak{g}_V M$ for a distribution V with maximal growth vector can be not locally trivial, if there exist nonisomorphic Lie algebras in $\widetilde{Nil}_N$.

Arbitrary algebra $\mathfrak{g} \in \widetilde{Nil}_N$ can be viewed on as an image of a free nilpotent k-step Lie algebra $\tilde{\mathfrak{g}}_k$ with respect to the homomorphism $\varphi : \tilde{\mathfrak{g}}_k \to \mathfrak{g}$ such that the diagram

$$\begin{array}{ccc} \tilde{\mathfrak{g}}_k & \xrightarrow{\varphi} & \mathfrak{g} \\ \Theta_k^{k-1}\downarrow & & \downarrow\Theta_k^{k-1} \\ \tilde{\mathfrak{g}}_{k-1} & \to & \mathfrak{g} \end{array}$$

is commutative. So in this case the manifold $\widetilde{Nil}_N$ can be identified with $H = \mathrm{Hom}\,(\mathbb{R}^{\hat{n}_k - n_{k-1}}, \mathbb{R}^{n - n_{k-1}})$ where $\hat{n}_k = \dim\ \tilde{\mathfrak{g}}_k$.

The set of all classes of isomorphic algebras from $\widetilde{Nil}_N$ can be identified with $H/GL_{n_1}\mathbb{R} \times GL_{n - n_{k-1}}\mathbb{R}$. So if

$$(3.1) \qquad n_1^2 + (n - n_{k-1})^2 = \dim\ GL_m\mathbb{R} < \dim H = (\tilde{n}_k - n_{k-1})(n - n_{k-1})$$

then nonisomorphic algebras exist.

All these considerations can be summarized in the following statement.

PROPOSITION 1.3.2. Assume (M, V) is a germ of a nonholonomic manifold, V is of maximal growth. Then:

a) In case the dimension of M is exact, then the bundle $\mathfrak{g}_V M$ is locally trivial.

b) In case the dimension of M is not exact, and the inequality (3.1) is true, then there exists a germ of a nonholonomic manifold (M, V) such that $n = \dim M$, $n_1 = \dim V$ and the bundle is not locally trivial.

Case (b) of this proposition is a most important case, in which the bundle $\mathfrak{g}_V M$ is not locally trivial.

Let's now turn to other cases, when distributions are not in general position, but the bundle $\mathfrak{g}_V M$ is locally trivial.

b) Nonholonomic Lie algebra

A tangent bundle over Lie group G is a direct product $G \times \mathfrak{g}$ of that group and its Lie algebra. Assume $(G, \mathfrak{v})$ is a nonholonomic Lie group, $(\mathfrak{g}, \mathfrak{v})$ is its nonholonomic Lie algebra. Let's define a functor ν, that puts into correspondence to each nonholonomic Lie algebra $(\mathfrak{g}, \mathfrak{v})$ its nilpotent homogeneous Lie algebra:

$$\sum_{i=1}^{k} \mathfrak{v}_i / \mathfrak{v}_{i-1}.$$

PROPOSITION 1.3.3. The bundle $\mathfrak{g}_V G$ of nilpotent Lie algebras over nonholonomic Lie group (G, V) is trivial

$$\mathfrak{g}_V G = G \times \nu(\mathfrak{g}, \mathfrak{v}).$$

Example. For arbitrary 3-dimensional Lie group G the bundle of nilpotent Lie algebras over G is isomorphic to $G \times n_3$ where n_3 is a 3-dimensional Heisenberg algebra.

c) Cases that can be (locally) reduced to group ones

Let's fix dimensions n of a manifold and n_1 of a distribution. We want to find out for which (n, n_1) the germ of a nonholonomic manifold $(\mathbb{R}^n, V)$, where V is a distribution in general position, is isomorphic to a germ of a nonholonomic Lie group.

The most important case (and at the same time the simplest of such cases) is that of contact manifolds (see, e.g. [28]). In this case the germ of $(\mathbb{R}^n, V)$ in every point is isomorphic to the germ of a nonholonomic Heisengerg group N_{2l+1}, where $2l + 1 = n$. Therefore the germ of a bundle $\mathfrak{g}_V \mathbb{R}^{2l+1}$ is trivial because for germ of contact manifolds: $\mathfrak{g}_V \mathbb{R}^{2l+1} = U \times \mathcal{N}_{2l+1}$ where U is a germ of $\mathbb{R}^{2l+1}$, $\mathcal{N}_{2l+1}$ $(2l+1)$-dimensional Heisenberg algebra.

1. Note that in the nongroup case globally the bundle $\mathfrak{g}_V M$ is generally speaking not trivial even if it is locally isomorphic to a group one, and the manifold M is contractible. The simplest examples of such situation are nonstandard contact structures on $\mathbb{R}^3$ (cf. [29]).

2. Besides contact structures, there exists only one case of dimensions $(n, n_1) = (4, 2)$ when the germ of a distribution in general position is isomorphic to a germ of left-invariant distribution on a Lie group (see [21]). In this case the germ $(\mathbb{R}^4, V)$ is isomorphic to a germ (G, V), where G is a nilpotent Lie group of degree 3 with 2 generators. Its Lie algebra is defined by the basis $\xi_1, \xi_2, \xi_3, \xi_4$ and the following relations $[\xi_1, \xi_2] = \xi_3$; $[\xi_1, \xi_3] = \xi_1$; $[\xi_2, \xi_3] = 0$; $[\xi_i, \xi_4] = 0$, $i = 1, 2, 3$.

(Distributions on this group were investigated as far back as in Engel's work – in connection with partial differential equations, (see [6]) (*).

4. ALGEBRAIC MODEL OF A JET OF A NONHOLONOMIC MANIFOLD

In the present point we describe a graded Lie algebra of formal vector fields, that serves as an algebraic model of Lie algebra of jets of vector fields on a nonholonomic manifold.

Let's consider Lie algebra $\mathcal{L}$ of formal vector fields, defined as a module over the ring $\mathbf{R} = \mathbb{R}\,[[x_1, \ldots, x_n]]$ with base $\{\partial/\partial x_i,\ i = 1, \ldots, n\}$. For growth vector $N = \{n_i\}_{i=1}^{k}$ let's define a function $\varphi_N : 2; n \to 1; k$ by setting $\varphi_N(j) = i$ for $n_{j-1} < i < n_j$. Let's define a graduation as the following:

$$\mathcal{M}^j = Lin\left\{ \sum_{i=1}^{n} x_i^{l_i} \mid \Sigma\, l_i\, \varphi_N(i) = j \right\}, \quad j = 1, 2, \ldots$$

The corresponding filtration in $\mathbf{R}$ is then defined by a chain of ideals

$$\mathcal{M}_j = \underset{i \geq j}{\oplus} \mathcal{M}^i .$$

(*) Nilpotentization arises many interesting problems concerning distributions generated by polynomial vector fields, some of them were discussed in [40].

The ring $\mathbb{R}$ with filtration, determined by a given growth vector, will be denoted by $\mathbb{R}_N$. The following statement describes the properties of this filtration.

PROPOSITIÒN 4.1.

1) $\mathbb{R}_N/\mathcal{M}_1 \simeq \mathbb{R}^1$

2) $\mathcal{M}_1/\mathcal{M}_2 \simeq \mathcal{M}^1 \simeq \mathbb{R}^{n_1}$

3) Denote by $\mathbb{R}_{(i)}$ the ring of formal power series of the variables $x_{n_{i-1}}, \ldots, x_{n_i}$, $\mathbb{R}_{(i)} = \mathbb{R}[[x_{n_{i-1}+1}, \ldots, x_{n_i}]]$ by $\overset{(i)}{\mathcal{M}}_1$ denote a maximal ideal in $\mathbb{R}_{(i)}$. Then the following equalities are true:

$$\mathbb{R}_N = \bigotimes_{i=1}^{k} \mathbb{R}_{(i)}$$

$$\mathcal{M}_l = \sum_{\{(i_1,\ldots,i_k) \mid \sum_{j=1}^{k} i_j \cdot j = l\}} \overset{(1)}{\mathcal{M}}{}_1^{i_1} \otimes \ldots \otimes \overset{(k)}{\mathcal{M}}{}_1^{i_k}$$

Let's define a function ord: $\mathbb{R}_N \to N$, as follows: for arbitrary $p \in \mathbb{R}_N$ ord $p = \min\{i \mid p \in \mathcal{M}_i\}$. We went to extend this function to the whole $\mathcal{L}$ thereby defining a filtration on $\mathcal{L}$ It is sufficient to extend ord to monomials of the type

$$\left(\prod_{i=1}^{n} x_i^{l_i}\right) \frac{\partial}{\partial x_j},$$

that generate $\mathcal{L}$ as a linear space. We define ord for such monomials by a formula

$$\operatorname{ord}\left(\prod_{i=1}^{n} x_i^{l_i}\right)\frac{\partial}{\partial x_j} \overset{\text{def}}{=} \operatorname{ord}\left(\prod_{i=1}^{n} x_i^{l_i}\right) - \varphi_N(j) =$$

$$= \left(\sum_{i=1}^{n} l_i \varphi_N(i)\right) - \varphi_N(j).$$

This extension of ord to $\mathcal{L}$ leads to a graduation $\mathcal{L}^j = Lin\{p \mid \operatorname{ord} p = j\}$ and a filtration

$$\mathcal{L}_j = \bigoplus_{i \geqslant j} \mathcal{L}^i$$

a decreasing chain of linear subspaces of an algebra $\mathcal{L}$. Let's denote a Lie algebra $\mathscr{L}$ with filtration, corresponding to growth vector N by $\mathscr{L}_N$.

PROPOSITION 4.2.

1) $\mathscr{L}_{-k} = \mathscr{L}_N$, $\mathscr{X}_{+\infty} = \bigcap_{i=-k}^{\infty} \mathscr{L}_i = 0$.

2) $\mathscr{L}_0$ is a subalgebra of the algebra $\mathscr{L}_N$

3) for every $i > 0$ $\mathscr{L}_i$ is on ideal in $\mathscr{L}_0$.

Filtrations $\mathscr{M}_j$ and $\mathscr{X}_j$ are consistent in the sense that $\mathscr{M}_i \mathscr{L}_j \subset \mathscr{L}_{i+j}$ and $\mathscr{L}_j \mathscr{M}_i \subset \mathscr{M}_{i+j}$ (in the first case multiplication means multiplying vector field and a power series, in the second case – differentiating of the power series along the vector field jets).

In our algebraic model free n_1-dimensional submodules of $\mathscr{L}$ play the role of regular distributions (see 1); in fact, as we'll see in the following point, it is sufficient to consider only the submodules of $\mathscr{L}_1$.

Let's describe properties of such submodules. First of all introduce the flag of a submodule $v : v_1 = v, \dots, v_i = [v_{i-1}, v_1]$ for $i \geqslant 1$. Define also the flag of linear spaces $V_1 \subset V_2 \subset \dots$ with $V_k = \mathbb{R}^n$, and

$$V_i = \mathbb{R}^{n_i} = Lin \left\{ \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_{n_i}} \right\}$$

PROPOSITION 4.3. Let $v \subset \mathscr{L}_{-1}$ be a free n_1-dimensional submodule. Then the following statements are true:

1) $v_i \subset \mathscr{L}_{-i}$ and $v_i / \mathscr{M}_1 = V_i$;

2) $v = \sum_{i=1}^{k} \mathscr{M}^i \hat{V}_i$ where $\hat{V}_i = Lin \left\{ \frac{\partial}{\partial x_{n_{i-1}+1}}, \dots, \frac{\partial}{\partial x_{n_i}} \right\}$

Arbitrary automorphism of the graded ring $\mathbb{R}_N$ (or of algebra $\mathscr{L}_N$) is uniquely defined by the linear transformation of $x_1, \dots, x_N$. Let's define an action of dilatation group $\{h_t\}$, $t \in \mathbb{R}^*$ on the ring $\mathbb{R}_N$ and Lie algebra $\mathscr{L}_N$:

$$h_t f(x_1, \dots, x_n) = f(tx_1, \dots, t^{\varphi(i)} x_i, \dots, t^k x_n)$$

$$h_t \left(f(x_1, \dots, x_n) \frac{\partial}{\partial x_j} \right) = f(tx_1, \dots, t^k x_n) \frac{\partial}{\partial x_j} \frac{1}{t^{\varphi(j)}}$$

The following proposition is evidently true.

PROPOSITION 4.3. The group $\{h_t\}$ acts as multiplication by t^j on every homogeneous component $\mathcal{M}^j$, $\mathcal{L}^j$.

Remark. The Lie algebra of formal vector fields, i.e. algebra of formal differential operators of first order, can be used to construct a universal enveloping algebra $\mathcal{U}_N$ of formal differential operators of arbitrary order over the ring $\mathbb{R}_N$ both the graduation and the action of group $\{h_t\}$ can be naturally extended to such an algebra.

5. QUASINORMAL FORM

A regular distribution V on a manifold M allows for arbitrary $x \in M$ to define a filtration on the ring of jets J_xM of smooth functions and in the Lie algebra of jets J_x *Vect* of vector fields over that ring. This filtration is defined by means of a function $\tilde{ord}$, whose construction is alike *ord* in the algebraic model (see 4). Let's define $\tilde{ord}$ consequently extending its domain of definition:

a) For jets of admissible vector fields assume $\tilde{ord}\,\xi = -1$

b) For vector field $\psi \in V_i \setminus V_{i-1}$ assume $ord\,\psi = -i$.

Definition of $\tilde{ord}$ for jets of functions from J_xM is based on the following statement.

PROPOSITION 5.1. Assume (M, V) is a nonholonomic manifold, $x \in M$. Then for arbitrary $f \in J_xM$ different from 0, there exist vector fields $\xi_1, \ldots, \xi_l \in J_x$ *Vect* such that

$$(\xi_l \cdot \ldots \cdot \xi_1 f)(x) \neq 0$$

c) For $f \in J_xM$, $f \neq 0$, define $\tilde{ord}$ as:

$$\tilde{ord}\ f = \min_{\{\xi_1, \ldots, \xi_l \mid (\xi_l, \ldots, \xi_1 f)(x) \neq 0\}} \left(- \sum_{i=1}^{l} \tilde{ord}\ \xi_i \right)$$

Assume $\tilde{ord}\ 0 = +\infty$.

The above defined function $\tilde{ord}$ determines a filtration $\mathcal{M}_1 \supset \mathcal{M}_2 \supset \ldots$ in the ring J_xM where the ideal

$$\mathcal{M}_i = \{ f \in J_xM \mid \tilde{ord}\ f \geqslant i \}.$$

d) For elements of Lie algebra J_x *Vect* the function $\tilde{ord}$ is defined as follows: for $\psi \in J_x$ *Vect*

$$\tilde{ord}\ \psi = \max \{ j \mid \forall_i\ \psi\, \mathcal{M}_i \subset \mathcal{M}_{i+j} \}$$

and the corresponding filtration is

$$\mathfrak{v}_j = \{\psi \in J_x \ Vect \mid \ \forall_i \ \psi \mathcal{M}_i \subset \mathcal{M}_{i+j}\}$$

Remark. Filtrations $\mathcal{M}_i$ and $\mathfrak{v}_i$ are uniquely determined by the correspondent graduations $\mathcal{M}^i$ and $\mathfrak{v}^i$. These graduations can be described directly, although such description is somewhat cumbersome. We'll now describe a quasinormal form theorem, that allows to obtain these direct descriptions by simple trasferring them from the algebraic model (see 4):

Evidently the rings R_N and $J_x M$ are isomorphic as abstract rings, and the Lie algebras $J_x \ Vect$ and $\mathcal{L}_N$ over that rings are isomorphic. Such as isomorphism is established by choosing a jet of coordinate system in the point $x \in M$. We want to probe that, moreover, one can always choose a special coordinate system, in which this isomorphisms of rings and algebras preserve filtration.

THEOREM 5.2. (on quasinormal form). Assume (M, V) is a germ of a nonholonomic manifold in some point x, N is a growth vector of V. Then for arbitrary isomorphism $\varkappa$ between R_N and $J_x M$, considered as abstract rings, there exist an automorphism $\psi : \mathrm{R}_N \circlearrowleft$ such that $\varkappa\psi$ is an isomorphism of filtered rings.

This theorem admits the following reformulation.

THEOREM 5.3. (on the quasinormal form of jets of nonholonomic distributions). Assume (M, V) is a germ of some nonholonomic manifold in a point x . Then there exists a coordinate system $\{ x_i : M \to \mathbb{R}^1, \ i = 1, \ldots, n \}$ in some neighbourhood of the point x, such that the jet of every admissible vector field lies in $\mathcal{L}_{-1}$.

We'll call such *coordinate systems consistent with the distribution* V. Note that generally speaking, there exist different coordinate systems, consistent with V.

COROLLARY [24]. Assume (M, V) is a germ of a nonholonomic manifold in some point x, and the coordinate system is consistent with V. Then the jet of arbitrary admissible vector field ξ has the following form:

$$\xi = \sum_S \mu_{i_0, \ldots, i_l, i_{l+1}} \prod_{j=1}^{l} x_{i_j} \frac{\partial}{\partial x_{i_{l+1}}}$$

where S contains only sets of indices for which the following inequality is true:

$$\varphi_V(i_{l+1}) \leqslant 1 + \sum_{j=1}^{l} \varphi_V(i_j).$$

We'll call such form of representation of admissible vector fields *quasinormal.* The prefix "quasi" notes that such representation is not unique.

Remark. Filtered isomorphism of rings and Lie algebras, that is induced by a coordinate system consistent with the distribution, allows to transfer to $J_x M$ and J_x *Vect* graduations from correspondingly $\mathbb{R}_N$ and $\mathcal{L}_N$. We'll denote the images of $\mathcal{L}^i$ that are obtained by this correspondence, also by $\mathcal{L}^i$. This correspondence allows also to define the action of quasihomogeneous dilatation group to $J_x M$ and J_x *Vect.*

6. PRINCIPAL QUASIHOMOGENEOUS PART (AND QUASIJETS) OF THE DISTRIBUTION

Assume (M, V) is a germ of some nonholonomic manifold, $N = \{n_i\}_{i=1}^k$ growth vector of the distribution V; $\mathcal{L}_N \simeq J_x$ *Vect* the Lie algebra of jets of vectorfields in some point $x \in M$. For $i \geqslant -k$ let us denote by p_i the projector

$$p_i : \mathcal{L}_N = \left(\bigoplus_{j=-k}^{\infty} \mathcal{L}^j \right) \to \mathcal{L}^i$$

For admissible vector field ξ $p_i \xi = 0$ if $i < -1$. By a principal quasihomogeneous part of an admissible vector field ξ we'll understand $\xi^{(-1)} = p_{-1}\xi$. Let us introduce projectors P_i, $i \geqslant -1$ where

$$P_i = \sum_{j=-1}^{i} p_j.$$

Let us define i-quasijets of admissible vector fields ξ as $J_V^i \xi = P_i \xi$. So the principal quasihomogeneous part of ξ is 0-quasijet of ξ. By i-quasijet $J_V^i V$ of a distribution V we'll understand the distribution $V^{(i)}$ generated by $\{J_V^i \xi\}$-i-quasijets of admissible vector fields; $V^{(0)}$ we'll call a principal quasihomogeneous part of a distribution V, i.e. 0-quasijet of V.

THEOREM 6.1. (on principal part of the distribution).

Assume (M, V) is a germ of a nonholonomic manifold in some point x and the distribution $V^{(0)}$ is a principal quasihomogeneous part (0-quasijet) of V. Then the germ of nonholonomic manifold $(M, V^{(0)})$ is isomorphic to the germ of osculating nonholonomic Lie group (G_x, V_x) (see 2).

Quasihomogeneous part of admissible vector field has the following form

$$\xi_{i_0}^{(-1)} = \frac{\partial}{\partial x_{i_0}} + \sum_{\{i_0,\ldots,i_{l+1}\}} \mu_{i_0,\ldots,i_l,i_{l+1}} \left(\prod_{j=1}^{l} x_{i_j}\right) \frac{\partial}{\partial x_{i_{l+1}}}$$

where

$$\varphi_V(i_{l+1}) = \sum_{j=0}^{l} \varphi_V(i_j).$$

If we choose base in the set of all admissible vector fields, then their principal quasihomogeneous parts form a basis for a Lie algebra $\mathfrak{g}_x$, and the coefficients $\mu_{i_0,\ldots,i_{l+1}}$ determine structural constants of that algebra. (The converse is not true: structural constants do not determine the coefficients $\mu_{i_0,\ldots,i_{l+1}}$ uniquely; this nonuniqueness can be easily seen on the simplest example – contact structure in $\mathbb{R}^3$). Therefore a question naturally arizes – how to choose a basis of vector fields, and a consistent coordinate system, so that the set of coefficients $\{\mu_{i_0,\ldots,i_l,i_{l+1}}\}$ be maximally simple. It turns out that in the most important case – that of distributions of maximal growth – the algebra $\mathfrak{g}_x$ is close to being free, and our aim can be achieved. Let's first describe the simplest case – the case of exact dimensions.

Distribution of maximal growth in exact dimensions

In this case algebra $\mathfrak{g}_x$ is the free nilpotent k-step Lie algebra $\tilde{\mathfrak{g}}_k$ with n_1 generators. Let's remind the classical description of linear basis in $\mathfrak{g}_k$ (see [26]).

Assume $\tilde{\mathscr{P}}$ is a free (noncommutative) monoid with n_1 generators; denote them by $\Theta_1, \ldots, \Theta_{n_1}$. There exists a natural graduation in this monoid: $\tilde{\mathscr{P}}^i$ is a set of all words of length i. Denote

$$\tilde{\mathscr{P}}_i = \bigcup_{j\leqslant i} \tilde{\mathscr{P}}^j.$$

Let's construct in $\tilde{\mathscr{P}}$ a subset $\mathscr{P}$ consequently defining components $\mathscr{P}^i = \mathscr{P} \cap \tilde{\mathscr{P}}_i$.

1) $\mathscr{P}^1$ is a set of generators $\Theta_1, \ldots, \Theta_{k_1}$. Let's order them arbitrarily, and denote this ordering by $>$.

2) $\mathscr{P}^2$ is a set of pairs ψ_1, ψ_2 where $\psi_1, \psi_2 \in \mathscr{P}^1$ and $\psi_1 < \psi_2$.

Let's then order arbitrarily all elements of $\mathscr{P}^2$, and assume that $\mathscr{P}^1 < \mathscr{P}^2$.

3) Suppose that $\mathscr{P}^1, \ldots, \mathscr{P}^i$, $i \geqslant 2$ are already constructed. Define $\mathscr{P}^{i+1}$ as subset of $\tilde{\mathscr{P}}^{i+1}$, formed by such words ψ, that admit the unique representation in the form $\psi = \psi_1(\psi_2\ \psi_3)$, where $\{\psi_i,\ i = 1, 2, 3\}$ satisfy the

following conditions:

a) $\psi_1 < (\psi_2\ \psi_3)$

b) $\psi_2 < \psi_3$

c) $\psi_1 \geqslant \psi_2$

Order $\mathscr{P}^{i+1}$ arbitrarily, and assume that $\mathscr{P}_i < \mathscr{P}^{i+1}$.

Such a set $\mathscr{P}$ is called a Hall family in the free monoid with n_1 generators. The choice of such family is not unique, because one can choose arbitrary ordering for each of the "homogeneous components" $\mathscr{P}^i$. For further considerations it will be convenient to suppose that some choice is fixed for every n_1.

Assume $g_1, \ldots, g_{n_1}$ is a set of generators of free Lie algebra $\tilde{\mathfrak{g}}$. A mapping $\nu : \Theta_i \to g_i$, $i = 1, \ldots, n_1$ can be extended to the mapping $\tilde{\mathscr{P}} \to \tilde{\mathfrak{g}}$, which transforms every word $\psi \in \mathscr{P}$ to the following elements of $\mathfrak{g}$: $\psi \in \mathscr{P}$ can be uniquely represented as a product of generators. Change every generator Θ_i in this product expression to g_i, and every round bracket to a Lie bracket.

THEOREM (Hall-Witt). The image of the subset $\mathscr{P}$ under the mapping ν is a homogeneous basis of the free Lie algebra $\mathfrak{g}$.

This theorem shows that it is convenient to enumerate the base of vector fields by the elements of Hall family; formula (6.1) shows that it also convenient to enumerate coordinates by the same elements. So we arrive at the following statement.

THEOREM 6.2. [24]. Assume (M, V) is a germ of a nonholonomic manifold in some point x, distribution V is of maximal growth $N = \{n_i\}_{i=1}^k$, and the dimension $n = \dim M$ is exact. Then there exists a coordinate system jet in x, consistant with the distribution, in which the principal quasihomogeneous parts of vector fields $\xi_{\Theta_1}, \ldots, \xi_{\Theta_{n_1}}$ forming the base for V have the following form:

$$\xi^{(-1)}_{\varkappa_0} = \sum_S x_{\varkappa_i} \cdot \ldots \cdot x_{\varkappa_l} \cdot \frac{\partial}{\partial x_{\varkappa_{l+1}}}$$

where the subset S includes all sets $\{\varkappa_i\}_{i=0}^{l+1}$, satisfying the condition

$$\varkappa_{l+1} = \varkappa_l(\varkappa_{l-1}(\ldots(\varkappa_1\ \varkappa_0)\ldots))$$

(here multiplication is understood in the sense of a free monoid).

Remarks. 1) For distributions of maximal growth the local trivial character of the bundle of nilpotent Lie algebras allows to obtain more precise results namely, one can obtain the canonic form not only for the principal quasihomogeneous part, but for the some other components as well, thus obtaining the normal form for jets of vector fields, see [24]

2) Theorem 6.2. can be extended to the case of distributions of maximal growth and non exact dimension of a manifold, only slight modifications are necessary, connected with the existence of nontrivial relations on the last step (see 3).

7. CARNOT-CARATHEODORY METRIC

By a metric tensor g_V on a nonholonomic manifold (M, V) will understand a positive definite symmetric quadratic form on the distribution V. If M is a Riemann manifold, then such a form is essentially obtained by restricting metric tensor to V, the inverse is also true, namely, every metric tensor on V can be extended to some metric tensor on TM.

A nonholonomic manifold, equipped with such a quadratic form on V will be called a *nonholonomic Riemann manifold.*

A curve $\gamma : \mathbb{R}^1 \to M$ is called admissible if it is a morphism of nonholonomic manifolds $(\mathbb{R}^1, T\mathbb{R}^1) \to (M, V)$. The existence of positive definite tensor on V allows to calculate lenghts of admissible curves.

THEOREM 7.1. (Rashevsky, Chow [30, 31]). Assume (M, V) is a nonholonomic Riemann manifold. Then arbitrary two points $x, y \in M$ can be connected by an admissible curve of finite length.

A metric ρ_V can be naturally defined on a nonholonomic Riemann manifold (M, V), in which a distance between the points $x, y \in M$ is defined as a minimal length of all admissible curves, connecting x and y :

$$\rho_V(x, y) = \inf_{S_{x,y}} \int_0^1 g_V(\dot\gamma(t), \dot\gamma(t))^{1/2}\, dt$$

where

$$S_{x,y} = \{\gamma : (\mathbb{R}^1, T\mathbb{R}^1) \to (M, V) \mid \gamma(0) = x,\ \gamma(1) = y\}.$$

This metric is called *Carnot-Caratheodory metric.*

By a *nonholonomic geodesic* we understand an admissible curve $\gamma : \mathbb{R}^1 \to M$ such that for arbitrary points x, y on γ that are sufficiently close to each other, the length of the interval of γ, connecting those points, equals to $\rho_V(x, y)$.

Filippov's lemma [32] allows to extend Hopf-Rinov theorem [33] to nonholonomic manifolds:

THEOREM 7.2. *For arbitrary two points x, y of a nonholonomic Riemann manifold there exists a shortest nonholonomic geodesic, connecting them.*

Remark. A specific feature of nonholonomic manifolds is that even for arbitrary close pairs of points there can exist several shortest geodesics; in other words the set of point, conjugate to some point $x \in M$, is not necessary isolated; it include submanifolds of positive dimensions, that contain that point x and its structure can be very complicated. The only thing one can guarantee that for any nonholonomic geodesic γ, γ is the unique shortest geodesic, connecting arbitrary sufficiently close points, belonging to γ.

Let's denote by $\mathscr{D}^V_\epsilon(x)$ the *ϵ-ball* in the sense of the *nonholonomic metric* ρ_V and by $S^V_\epsilon(x)$ – the ϵ-sphere. The following statement, establishes equivalence of different nonholonomic Riemann metrics.

PROPOSITION 7.3. Assume (M, V) is a germ of a nonholonomic manifold. Then arbitrary two nonholonomic Riemann metrics on (M, V) are equivalent.

Metric on homogeneous nilpotent Lie groups (G, V) are particular cases of nonholonomic metrics. Every nonholonomic metric on such a group is generated by a standard distribution on G and due to the fact that all standard distributions belong to one orbit with respect to the action of automorphisms group Aut G, arbitrary two nonholonomic metrics on G can be transformed into each other by some automorphism. Let's describe the properties of nonholonomic metrics on such groups.

PROPOSITION 8.4. Assume (G, V) is a nonholonomic nilpotent homogeneous Lie group; ρ_V is a nonholonomic left-invariant metric on (G, V), $\{H_t\}$ – group of quasihomogeneous dilatation, acting on G. Then the following statements are true:

1) For arbitrary nonholonomic geodesic γ on G, $H_t\gamma$ is also a nonholonomic geodesic.

2) For arbitrary two points $x, y \in G$

$$\rho_V(H_t x, H_t y) = t\rho_V(x, y)$$

3) Balls $\mathscr{D}^V_\epsilon(x)$ (and correspondingly spheres $S^V_\epsilon(x)$) form "quasihomogeneous" families of sets, i.e.

$$H_t \mathscr{D}^V_\epsilon(x) = \mathscr{D}^V_{t\epsilon}(H_t x)$$

and

$$H_t\ S_\epsilon^V(x) = S_{t\epsilon}^V(H_t x)$$

8. APPROXIMATION OF A GERM OF A NONHOLONOMIC RIEMANN MANIFOLD

Assume M is a Riemann manifold, V and $\tilde{V}$ are two germs of regular distributions in the point $x \in M$. We'll say that the germs V and $\tilde{V}$ have tangency of order j in the point x, if $V^{(i)} = \tilde{V}^{(i)}$ i.e. $J_V^i V = J_V^i \tilde{V}$. The tangency of distributions V and $\tilde{V}$ lead to closeness of the correspondent nonholonomic metrics ρ_V and $\rho_{\tilde{V}}$.

PROPOSITION 8.1. Assume M is a germ of a Riemann manifold, $x \in M$; V and $\tilde{V}$ are germs of regular and completely nonholonomic distributions, that have tangency of order $j \geqslant 0$ in the point x.

Then the following inclusions are true:

$$(1 - O(\epsilon^{j+1}))\,\mathscr{D}_\epsilon^{\tilde{V}}(x) \subset \mathscr{D}_\epsilon^{V}(x) \subset (1 + O(\epsilon^{j+1}))\ \mathscr{D}_\epsilon^{\tilde{V}}(x).$$

Here by a multiplication of a ball and a number we understand the multiplication in the sense of linear structure, generated by arbitrary map on M. The above given asymptotics does not depend on the choice of the map, and is exact.

Due to the fact that every regular and completely nonholonomic distribution V has tangency of order O with its principal quasihomogeneous part $V^{(0)}$, we obtain the following statement.

COROLLARY. Assume (M, V) is a germ of a nonholonomic Riemann manifold in some point x; $V^{(0)}$ is a principal quasihomogeneous part of V. Then the following inclusions are true:

$$(1 - O(\epsilon))\ \mathscr{D}_\epsilon^{V^{(0)}}(x) \subset \mathscr{D}_\epsilon^{V}(x) \subset (1 + O(\epsilon))\ \mathscr{D}_\epsilon^{V^{(0)}}(x)$$

The germ of a nonholonomic manifold $(M, V^{(0)})$ in a point x can be identified with the germ of a nonholonomic nilpotent Lie group (G_x, V_x) (see 6). Nonholonomic metric on (M, V) induces left-invariant Carnot-Caratheodory metric on the osculating Lie group (G_x, V_x). Together with the statement that Carnot-Caratheodory metric on (G_x, V_x) is quasihomogeneous, this leads to the following theorem.

THEOREM 8.2. Assume (M, V) is a germ of a nonholonomic Riemann manifold in some point x; (G_x, V_x) is as osculating nonholonomic nilpotent Lie group, $\mathscr{D}$ – a unit ball with the center in the identity element of G (in the sense of Carnot-Caratheodory metric). Them the following inclusions are true:

$$(1 - O(\epsilon))\, H_\epsilon \mathscr{D} \subset \mathscr{D}^V_\epsilon(x) \subset (1 + O(\epsilon))\, H_\epsilon \mathscr{D}$$

Remark. Calculating the ball $\mathscr{D}$ on a nilpotent homogeneous Lie group is in itself an interesting non-trivial problem. For the simplest case that is three dimensional Heisenberg group the precise form of $\mathscr{D}$ is given in [20].

The above theorem together with the following statement allows to obtain an estimate of the ϵ-ball $\mathscr{D}^V_\epsilon(x)$ in the sense of Carnot-Caratheodory metrics.

PROPOSITION 8.3. A unit ball in the sense of Carnot-Caratheodory metric has a non empty interior in the Riemann topology (i.e. topologies, generated by the original Riemann metric and by Carnot-Caratheodory metric, coincide).

Let's give an estimate for $\mathscr{D}^V_\epsilon(x)$ for small ϵ. Assume $\{x_i : M \to \mathbb{R}^1,\ i = 1, \ldots, n\}$ is a coordinate system in some neighbourhood of the point x. Define a parallelepiped $\Pi_{c,\epsilon}(x)$ for $c > 0$ and sufficiently small positive ϵ by a formula

$$\Pi_{C,\epsilon}(x) = \{ y \in M \mid \ |x_i(y) - x_i(x)| \leqslant c\, \epsilon^{\varphi_V(i)} \}$$

THEOREM 8.4. (On a parallelepiped [23]). Assume (M, V) is a germ of a nonholonomic Riemann manifold in a point x, $\{x_i\}_{i=1}^n$ is a coordinate system on M consistent with the distribution V. Then there exist $c > 0$ and $C > 0$ such that for any sufficiently small positive ϵ the following inclusions take place:

$$\Pi_{c,\epsilon}(x) \subset \mathscr{D}^V_\epsilon(x) \subset \Pi_{C,\epsilon}(x)$$

9. HAUSDORFF DIMENSION OF NONHOLONOMIC MANIFOLD

Assume G is a nilpotent homogeneous k-step Lie group, V a standard distribution on G, $\{n_i\}_{i=1}^k$ – growth vector of V. By a *homogeneous dimension* we'll understand a value $d_V G$, defined by a formula

$$d_V G = \sum_{j=1}^{k} j\,(n_j - n_{j-1})$$

Let's present an interpretation of this notion. Assume a Haar measure is defined on the group G. Then, as the following statement shows, homogeneous dimension is an exponent, expressing the change of volume in the process of dilatation.

PROPOSITION 9.1. Assume (G, V) is a nonholonomic nilpotent homogeneous

Lie group, μ-Haar measure on G, $X \subset G$. Then

$$\mu(H_t X) = t^{d_V} \cdot \mu(x)$$

The quasihomogeneous character of the family of nonholonomic balls (in the sense of Carnot-Caratheodory metric) allows to interpret $d_V G$ as a Hausdorff dimension of the metric space (G, ρ_V).

PROPOSITION 9.2. [10]. Hausdorff dimension of a nonholonomic nilpotent Lie group (G, ρ_V) coincides with its homogeneous dimension.

The above-mentioned results about the possibility to approximate a germ of Carnot-Caratheodory metric on a nonholonomic manifold by a metric on the germ of osculating Lie group lead to the conlcusion that the corresponding Hausdorff dimensions coincide. So we have obtained the following statement.

THEOREM 9.3. Assume (M, V) is a nonholonomic manifold; $\{n_i\}_{i=1}^{k}$ – is a growth vector of the distribution V. Then for every Riemann metric on M Hausdorff dimension $d_V M$ of a nonholonomic manifold (M, V) is given by the following formula

$$d_V M = \sum_{j=1}^{k} j(n_j - n_{j-1}).$$

10. FOUNDATIONS OF NILPOTENT ANALYSIS

Main notions of analysis on nonholonomic manifolds can be naturally formulated in terms of a bundle of nilpotent Lie algebras. In the present paragraph we'll discuss the simplest notions – those of first and second differential, Hessian and critical point.

By a *differential* d_φ of a mapping $\varphi : (M, V) \to (\tilde{M}, \tilde{V})$ of nonholonomic manifolds we understand a morphism of bundle of nilpotent Lie algebras $\mathfrak{g}_V M \to \mathfrak{g}_{\tilde{V}} \tilde{M}$ (this definition was introduced in 3). In particular, a differential of a function f on a nonholonomic manifold: $(M, V) \to (\mathbb{R}^1, T\mathbb{R}^1)$ is a morphism of a bundle of Lie algebras $\mathfrak{g}_V M$ into the trivial bundle $\mathbb{R}^1 \times \mathbb{R}^1$ of commutative 1-dimensional Lie algebras. Therefore the morphism df annihilates the bundle of commutants $[\mathfrak{g}_V M, \mathfrak{g}_V M]$ and hence determines a mapping

$$\widetilde{df} : V \simeq \mathfrak{g}_V M / [\mathfrak{g}_V M, \mathfrak{g}_V M] \to R^1$$

So we obtain the following statement.

PROPOSITION 10.1. The set of differentials of functions on a nonholonomic manifold (M, V) is naturally isomorphic to the codistribution V^*.

Another (equivalent) way to construct the space of differential is to interpret them as elements of $\mathcal{M}_1/\mathcal{M}_2$; this interpretation leads to the same result, because

$$\mathcal{M}_1/\mathcal{M}_2 \simeq V^*$$

Assume f is a function on a nonholonomic manifold (M, V). A point $x \in M$ is called a critical point for f, if $df(x) = 0$. This definition of a critical point can be evidently reformulated.

PROPOSITION 10.2. A point x is a critical point of a function f on a nonholonomic manifold iff x is a critical point for the restriction of f to arbitrary admissible curve, containing x.

In the critical point x of the mappint $(M, V) \to (\mathbb{R}^1, T\mathbb{R}^1)$ of nonholonomic manifolds one can define the second differential as follows. Choose some neighbourhood U of a point $x \in M$, and choose a coordinate system in U, that is consistent with the distribution V. Define d^2f:

$$d^2f(y) = \lim_{t\to 0} \frac{f(H_t y)}{t^2}$$

where H_t is the dilatation.

PROPOSITION 10.3. 1) The second differential of a function is correctly defined for arbitrary function, for which x is a critical point, i.e. for $f \subset \mathcal{M}_2^x$ the equality $d^2f = 0$ is equivalent to the inclusion $f \in \mathcal{M}_3^x$.

2) The space of second differentials of all functions, for which the point x is critical, can be naturally identified with $\mathcal{M}_2^x/\mathcal{M}_3^x$.

Arbitrary element of $\mathcal{M}_2^x/\mathcal{M}_3^x$ can be considered as a sum of a quadratic form on V^* and an element of $(V_2/V_1)^*$. In case we use a coordinate system, consistent with the distribution V, in the neighbourhood of the critical point $x \in (M, V)$ then in that point d^2f equals to the sum of a quadratic from of the coordinates $x_1, \ldots, x_{n_1}$ and the linear form of the coordinates x_{n_1+1} $\ldots, x_{n_2}$.

We want to define a *Hessian* of a function f on a nonholonomic manifold (M, V). It is natural to say that a Hessian is positive definite if the second derivative of f along arbitrary admissible vector field is positive. Assume $\{\xi_i\}_{i=1}^{n_1}$ is a basic of module of admissible vector fields. Define

$$\xi = \sum_{i=1}^{n_1} \alpha_i \xi_i$$

Due to the fact that the point x is critical, we obtain $\xi_i f = 0$.
The condition that $\xi^2 f > 0$ for arbitrary $\{\alpha_i\}_{i=1}^{n_1}$ leads to

$$\sum_{i,j=1}^{n_1} (\xi_i \xi_j f) \alpha_i \alpha_j > 0.$$

This means that the symmetric matrix $A_{ij} = 1/2\ (\xi_i \xi_j + \xi_j \xi_i) f(x)$ is positive definite. These arguments motivate the following definition.

DEFINITION. Assume f is a function, defined on a germ (in point x) of some nonholonomic manifold (M, V); x is a critical point of f; $\xi_1, \ldots, \xi_{n_1}$ is a basic of the module of admissible vector fields. By a Hessian of a function f we understand the following $n_1 \times n_1$ matrix

$$A_{ij} = \frac{1}{2}(\xi_i \xi_j + \xi_j \xi_i) f = \left(\xi_i \xi_j - \frac{1}{2}[\xi_i, \xi_j]\right) f.$$

PROPOSITION 10.4. A Hessian of the function f defined on a nonholonomic manifold (M, V), in its a critical point x, does not depend on the choice of a basis of the admissible vector fields.

A critical point of a function $f: (M, V) \to (\mathbb{R}^1, T\mathbb{R}^1)$ is called nondegenerate, if its Hessian is nondegenerate. Alike classical Morse theory, one can introduce the notion of an index of a critical point. The natural further development of this paragraph should be the construction of nonholonomic Morse theory. However, the size of the present survey does not allow that, and this theory will be the subject of a separate publication.

Part II Nonholonomic Riemann geometry and differential operators

11. HAMILTON AND LAGRANGE FORMALISMS MIXED BUNDLE

Assume (M, V) is a nonholonomic Riemann manifold; we suppose that the metric tensor g_V is extended to the whole tangent bundle. By a Hamiltonian H on (M, V) we understand a 2-tensor on TM defined by the formula

$$H(v) = g(v, v) \qquad \text{for } v \in V$$

$$H(v) = 0 \qquad \text{for } v \in V^{\perp}$$

Identifying TM and T^*M, we can assume that H is a 2-tensor on T^*M and consider a Hamilton system (T^*M, Ω, H) where Ω is a standard symplectic structure on T^*M.

PROPOSITION 11.1. (Hamilton formulation). The corresponding dynamical system with a degenerate Hamiltonian H determines a nonholonomic geodesic $\mathcal{N}_g$ flow on T^*M.

Remark. This $\mathcal{N} g$ flow does not depend on the concrete extension of metric tensor to TM.

Let's now turn to a description of Lagrange formalism.

Centaurus. A nonholonomic geodesic is a solution of a conditional variational problem with nonholonomic constraints, expressed by V. Due to the fact that arbitrary two points of a nonholonomic manifold (M, V) can be connected by a nonholonomic geodesic, the initial data for such a geodesic must include $n = \dim M$ parateters; $n_1 = \dim V$ parameters correspond to the initial admissible velocity vector $v \in V(x)$, one have to interpret the other $(n - n_1)$ parameters; due to $(n - n_1) = \dim V^\perp$ it is natural to interpret them as a 1-form $\omega \in V^\perp \subset T^*M$. This 1-form is an invariant way to write down Lagrange multipliers ("corresponding" to the conditional variational problem). Therefore in order to describe the nonholonomic geodesic flow we must consider the direct sum $V \oplus V^\perp$ of the distribution $V \subset TM$ and its annihilator $V^\perp \subset T^*M$. We'll call this sum a *mixed bundle*, or a *centaurus.*

PROPOSITION 11.2 (Lagrange formulation). Euler-Lagrange equations of the conditional variational problem on a nonholonomic manifold determine a flow on the mixed bundle $V \oplus V^\perp$. Nonholonomic geodesic equations are of the following form:

$$\nabla_\gamma \dot\gamma = \frac{\partial}{\partial t}\, \omega - \dot\gamma \,\lrcorner\, d\omega$$

(Here $\dot\gamma \,\lrcorner\, d\omega$ is a 1-form such that for vector field ξ: $(\dot\gamma \,\lrcorner\, d\omega)\, \xi = \omega(\dot\gamma, \xi)$.

Classical Legendre transformations for a nondegenerate quadratic Hamiltonian allow to identity tangent and cotangent bundles. In our case we can identify T^*M and $V \oplus V^\perp$ as follows: $V^\perp \subset T^*M$, and $(V^\perp)^\perp$ and V can be identified by means of Legendre transformation. Here we can again see that the $\mathcal{N} g$ – flow is uniquely determined by the metric on V (independently on its extension).

The advantage of Hamilton approach to nonholonomic case is that all usual Hamilton structures remain true (reduction, a notion of integrable and so on). At the same time its drawback (that is closely connected with its advantage) is that the specific features of nonholonomity are masked by the possibility to consider degenerate Hamiltonians, so that it is not possible to distinguish between integrable and completely nonholonomic distributions – while Lagrange formalism distinguishes these cases.

Lagrange ("centaurus") approach is more closely connected with classical Lagrange method, it is more geometrical (e.g. this language is more convenient for discussing questions, connected with conjugate points, wave fronts etc., see 15 and [17]) .

12. NONHOLONOMIC GEODESIC EQUATIONS IN ORTHOGONAL FRAME

Assume (M, V) is a germ of a nonholonomic manifold in a point x. Let's choose the following basis of n vector fields on M:

1) assume $\xi_1, \ldots, \xi_{n_1}$ is an orthonormed basis for the distribution V.

2) assume $\xi_1, \ldots, \xi_{n_i}$ are already constructed, and form an orthonormed basis for V_i. Complete this list by adding vector fields $\xi_{n_i+1}, \ldots, \xi_{n_{i+1}}$ so that we obtain the orthonormed basis for V_{i+1}.

We'll call such a basis *orthogonal frame* of a nonholonomic manifold.

Let's choose a basis $\{\omega_i\}_{i=1}^n$ of 1-forms, dual to $\{\xi_i\}_{i=1}^n$.

Assume $\dot{\gamma}$ is a nonholonomic geodesic, then the admissible vector γ can be represented in the form

$$\dot{\gamma} = \sum_{i=1}^{n_1} v_i \xi_i$$

any 1-form ω from $V^\perp$ is a linear combination of $\omega_{n_1+1}, \ldots, \omega_n$.

By C_{ij}^l we denote structural functions of the set of vector fields $\{\xi_i\}$:

$$C_{ij}^l = \langle [\xi_i, \xi_j], \ \xi_l \rangle$$

(C_{ij}^l is a function on the manifold M). By Γ_{ij}^l we denote Kristoffel symbols, determined by a Riemann metric on M:

$$\Gamma_{ij}^l = \langle \nabla_{\xi_j} \xi_i \ \ \xi_l \rangle$$

In the orthogonal frame $\xi_1, \ldots, \xi_n$ the nonholonomic geodesic equations take, the following form.

PROPOSITION 12.1. Assume (M, V) is a germ of a nonholonomic manifold in some point x, $\{\xi_i\}_{i=1}^n$ an orthogonal frame on (M, V). Then nonholonomic geodesics are determined by the following equations

$$
(12.1)\qquad \begin{cases}
\dot{\gamma} = \sum_{i=1}^{n_1} v_i \xi_i \\
\dot{v}_i = \sum_{\substack{1 \leq l \leq n_1 \\ n_1 < j \leq n}} C_{il}^j v_l \lambda_j - \sum_{1 \leq l,j \leq n_1} \Gamma_{ij}^l v_l v_j \quad i = 1, \ldots, n_1 \\
\dot{\lambda}_i = \sum_{\substack{1 \leq l \leq n_1 \\ n_1 < j \leq n}} C_{il}^j v_l \lambda_j + \sum_{1 \leq j,l \leq n_1} \Gamma_{ij}^l v_l v_j \quad i = n_1 + 1, \ldots, n
\end{cases}
$$

For most interesting cases (for distributions of maximal growth, for Lie groups, especially for nilpotent homogeneous Lie groups) nonholonomic geodesic equations are essentially simpler. For the above-mentioned cases the simplified equations are presented in [17]. Here we consider only the case of nilpotent homogeneous Lie groups.

PROPOSITION 12.2. Assume (G, V) is a germ of a nilpotent nonholonomic Lie group, $\{\tilde{\xi}_i\}_{i=1}^n$ is an orthogonal left-invariant frame on this nonholonomic Lie group, $\tilde{C}_{ij}^l$ – its structural constants. Then the nonholonomic geodesic γ on (G, V) is determined by the following system of equations:

$$
(12.2)\qquad \begin{cases}
\dot{\tilde{\gamma}} = \sum_{i=1}^{n_1} \tilde{v}_i \tilde{\xi}_i; \\
\dot{\tilde{v}}_i = \sum_{\substack{1 \leq l \leq n_1 \\ n_1 < j \leq n}} \tilde{C}_{il}^j \tilde{v}_l \tilde{\lambda}_l, \qquad i = 1, \ldots, n_1; \\
\dot{\tilde{\lambda}}_i = \sum_{\substack{1 \leq l \leq n_1 \\ \varphi(j) = \varphi(i) + 1}} \tilde{C}_{il}^j \tilde{v}_l \tilde{\lambda}_j \quad i = n_1 + 1, \ldots, n.
\end{cases}
$$

As we have already seen (in 8), the germ of a nonholonomic manifold can be approximated by the germ $(M, V^{(0)})$, that, in its turn, is isomorphic to the germ of osculating nonholonomic nilpotent Lie group (G_x, V_x). Let's describe the connection between nonholonomic geodesic equations for (M, V) and for (G_x, V_x).

If $\{\xi_i\}$ is an orthogonal frame on (M, V) then main quasihomogeneous parts $\{\xi_i^{(-1)}\}$ of its elements form a left-invariant orthogonal frame on the germ of the nonholonomic Lie group $(M, V^{(0)}) \simeq (G_x\ \ V_x)$. Structural constants $\widetilde{C}^l_{ij}$ of the frame $\{\xi_i^{(-1)}\}$ are connected with the structural functions $C^l_{ij}(y)$ of the original frame $\{\xi_i\}$ as follows:

$$\tilde{c}^l_{ij} = \begin{cases} c^l_{ij}(x), & \text{if } \varphi_V(l) = \varphi_V(i) + \varphi_V(j) \\ 0, & \text{else} \end{cases} \tag{13.3}$$

(if $\varphi_V(l) > \varphi_V(i) + \varphi_V(j)$ then $c^l_{ij} \equiv 0$ and $\tilde{c}^l_{ij} = 0$).

Remark that the expression Kristoffel symbols in terms of structural constants show that

$$\widetilde{\Gamma}^l_{ij} = \frac{1}{2}(\widetilde{C}^l_{ij} + \widetilde{C}^i_{lj} + \widetilde{C}^j_{li}) = 0$$

for $\{\xi_i^{(-1)}\}$ frame.

Let's call the procedure of transformation structure function C^l_{ij} to structural constant $\widetilde{C}^l_{ij}$ by using the formula (13.3) a *"freezing of coefficients"* procedure.

PROPOSITION 12.3 (on freezing of coefficients). Assume (M, V) is a germ in point x of a nonholonomic manifold, $V^{(0)}$ is a germ of the principal part of V in the same point. Then the equations of nonholonomic geodesics on $(M, V^{(0)}) \simeq (G_x, V_x)$ (osculating Lie group) can be obtained from the equations for (M, V) by a "freezing of coefficients" procedure.

The geodesic equations on (M, V) and $(M, V^{(0)})$ are close to each other in the metrical sense:

PROPOSITION 12.4. Assume γ and $\tilde{\gamma}$ are nonholonomic geodesics correspondingly for the germs (M, V) and $(M, V^{(0)})$ of nonholonomic manifolds, and γ and $\tilde{\gamma}$ have the same initial data $(\{v_i\}_{i=1}^{n_1}, \{\lambda_j\}_{j=n_1+1}^{n})$, and $\{x_i\}_{i=1}^{h}$ is a coordinate system on M consistent with the distribution V. Then the following asymptotics is true for $\epsilon \to 0$

$$|x_i(\gamma(\epsilon)) - x_i(\tilde{\gamma}(\epsilon))| = 0\,(\epsilon^{\varphi_V(i)+1}), \qquad i = 1, \ldots, n.$$

13. REDUCTION THE $\mathcal{N}g$-FLOW ON HEISENBERG GROUP

Like a geodesic flow on the Lie group G, the $\mathcal{N}g$-flow on G admits reduction, i.e. it is a skew product. Indeed, the mixed bundle over the nonholonomic Lie group (G, V) is a direct product $G \times (v \oplus v^\perp)$, where v is a linear subspace of the Lie algebra $\mathfrak{g}$, determining the left-invariant distribution V, and $v^\perp$ is the annihilator of subspace v in a coalgebra $\mathfrak{g}^*$. In this case every orthogonal frame $\{\xi_i\}$ for Lie group is induced by orthogonal frame $\{\tilde{\xi}_i\}$ in the nonholonomic Lie algebra, and due to the fact that for Lie groups the last two groups of equations in (13.1) have constant coefficients, they determine the flow on $v \oplus v^\perp$.

PROPOSITION 13.1 (on reduction of $\mathcal{N}g$-flows). Assume (G, V) is a nonholonomic Lie group. Then a $\mathcal{N}g$-flow on its mixed bundle $G \times (v \oplus v^\perp)$ is a skew product with base $v \oplus v^\perp$ and band G. The flow on the base is determined by the last two groups of equations from (13.1), the flow in the band – by the equation

$$\gamma^{-1}\dot{\gamma} = \sum_{i=1}^{n_1} v_i \xi_i .$$

The flow on the base $v \oplus v^\perp$ posesses an integral, namely energy integral $\langle \dot{\gamma}, \dot{\gamma} \rangle = \text{const}$, i.e.

$$\sum_{i=1}^{n_1} v_i^2 = \text{const}.$$

This integral allows to select the family of invariant "cylinders" $S_\delta \times v^\perp \subset v \times v^\perp$.

Let's turn to consideration the simplest example of $\mathcal{N}g$-flow – a flow on the 3-dimensional Heisenberg group, choose an orthogonal frame in the 3-dimensional Heisenberg algebra $\mathcal{N}_3$: ξ_1, ξ_2, ξ_3, where ξ_3 lies in the center of $\mathcal{N}_3$, $v = Lin\{\xi_1, \xi_2\}$ and $\xi_3 = [\xi_1, \xi_2]$. It is possible to choose such metric, for which this frame is orthogonal, because any other metric can be transformed to this one by means of some automorphism of the group (see 8). Energy conservation leads to $v_1^2 + v_2^2 = \text{const}$.

Changing (if necessary) parametrization on nonholonomic geodesics, we can assume, that tyis constant equals to 1. So we can introduce a new parameter φ so that $v_1 = \cos\varphi$, $v_2 = \sin\varphi$. The equations on the level surface of energy interval $S^1 \times \mathbb{R}^1 = \{(\varphi, \lambda)\}$ take the form

$$\begin{cases} \dot{\varphi} = \lambda \\ \dot{\lambda} = 0. \end{cases}$$

So projections of nonholonomic geodesics on the base of the fibre bundle are either circles $v_1 = \cos(\varphi + \varphi_0)$, $v_2 = \sin(\varphi + \varphi_0)$ or points $v_1 = c_1, v_2 = c_2$. We'll call them correspondingly geodesics of I and II type.

Heisenberg group, when viewed on as a topological space, coincides with $\mathbb{R}^3$. One can choose the orthogonal frame, consisting of the following vector fields:

$$\xi_1 = \frac{\partial}{\partial x_1} + \frac{x_2}{2}\frac{\partial}{\partial x_3}, \quad \xi_2 = \frac{\partial}{\partial x_2} - \frac{x_1}{2}\frac{\partial}{\partial x_3}, \quad \xi_3 = \frac{\partial}{\partial x_3}.$$

Second type geodesics are left shift of one-parametric subgroup with admissible generators of Heisenberg group.

The equation in the band for I-type geodesic:

$$\gamma^{-1}\dot{\gamma} = \cos(\lambda t + \varphi_0)\,\xi_1 + \sin(\lambda t + \varphi_0)\,\xi_2$$

can also be simply solved, and thus we obtain the following description of a $\mathcal{N}g$-flow on Heisenberg group.

PROPOSITION 13.3. For the nonholonomic Heisenberg group geodesics of first type are cylindric spirals, whose axes are arbitrary left shifts of the froup center. Geodesic of second type are left shifts of 1-parametric subgroups with admissible generators.

The $\mathcal{N}g$-flow on compact homogeneous spaces of Heisenberg group – is a somewhat more interesting object. Let's present a qualitative description of such flows.

PROPOSITION 13.3. There are two classes of nonholonomic geodesics on a compact homogeneous space of a 3-dimensional Heisenberg groups:

1) curves, everywhere dense in 2-dimensional tors;

2) circles.

The set of initial data of the geodesics of I-class has measure 1, initial data corresponding to II class are everywhere dense.

In [17] geodesic flows on compact homogeneous spaces of N_3 is described in more detail. The same paper contains a description of $\mathcal{N}g$-flows for all 3-dimensional Lie groups and their compact homogeneous spaces. Here we'll analyses only $\mathcal{N}g$-flow on the compact homogeneous space of group $SL_2\mathbb{R}$. The reason is that this case turns out to be the most interesting – just like in the classical case.

14. NONHOLONOMIC GEODESIC FLOW ON $SL_2\mathbb{R}$. CONNECTION WITH THE CLASSICAL GEODESIC FLOW

In classical (holonomic) case the most interesting of geodesic flows are the flows on compact manifolds of negative curvature. The simplest example of such manifold is a homogeneous space of $SL_2\mathbb{R}$.

Choose the following basis η_1, η_2, η_3 in the Lie algebra $sl_2\mathbb{R}$.

$$\eta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Arbitrary nonholonomic 2-dimensional left-invariant distribution is generated by a plane $v \subset sl_2\mathbb{R}$ such that $v + [v, v] = sl_2\mathbb{R}$. We'll call such a plane nonholonomic.

PROPOSITION 14.1. Arbitrary nonholomorphic plane $v \subset sl_2\mathbb{R}$ can be transformed by an automorphism of Lie algebra $sl_2\mathbb{R}$ into one of the following two planes:

a) orthohyperbolic plane

$$v^1 = Lin(\eta_1 + \eta_2, \eta_1 - \eta_2) = \left\{ \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \middle| \alpha, \beta \in R \right\}$$

b) orthoelliptic plane

$$v^2 = Lin(\eta_3, \eta_1) = \left\{ \begin{pmatrix} \alpha & \beta \\ \beta & -\alpha \end{pmatrix} \middle| \alpha, \beta \in R \right\}$$

Let's call the basis $\{\eta_1 + \eta_2, \eta_1 - \eta_2\}$ of the plane v^1 (and the basis $\{\eta_3, \eta_1\}$ of the plane v^2) appointed basis. Let's now describe metric tensors on nonholonomic planes in Lie algebra $sl_2\mathbb{R}$.

PROPOSITION 14.2. The orbits of metric tensors on v^1 (v^2) with respect to automorphisms group of a nonholonomic algebra Aut $(sl_2\mathbb{R}, v^1)$ (resp. Aut $(sl_2\mathbb{R}, v^2)$) can be parametrized by positive reals $\mathbb{R}_+$. For arbitrary $m > 0$ this orbit contains the tensor g_m, whose matrix in the appointed basis is $\begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix}$

Detailed investigation of $\mathcal{N}g$-flows on all compact homogeneous spaces of $SL_2\mathbb{R}$ is held in [19]. Here we'll describe only one of such cases. We'll consider only restrictions of the $\mathcal{N}g$-flow to the invariant set, that is singled out by the condition $\|\dot{\gamma}\| = 1$ on nonholonomic geodesic (see 13). This set is a bundle with base $S^1 \times \mathbb{R}^1$ and band $SL_2\mathbb{R}$.

THEOREM 14.3. Assume $\mathcal{D}$ is a compact homogeneous space of Lie group $SL_2\mathbb{R}$, and a metric tensor g_m, $m \neq 1$, is given. Then:

1) All the trajectories are closed on the cylinder $S^1 \times \mathbb{R}^1$ (except two fixed points and four separatrices, connecting those points),

2) almost all ergodic components of a $\mathcal{N}g$-flow are 4-dimensional manifolds;

every such component is $\alpha \times \mathscr{D}$, where α is a closed curve belonging to a base $S^1 \times \mathbb{R}^1$.

Let's connect this $\mathscr{N}g$-flow with a well-known geodesic flow on the planes of constant negative curvature. Assume is a 4-dimensional ergodic component of the $\mathscr{N}g$-flow. This flow determines a 1-parametric subgroup $\{T_t\}$ on $\alpha \times \mathscr{D}$. Denote by τ the period of the closed curve α. and consider the cascade $\{T_{n\tau}\}_{n=1}^{\infty}$. For any $x \in \alpha$ the set $x \times \mathscr{D}$, are invariant components with respect to this cascade. On each of this components the mapping T_τ coincides with the shift Lg of a homogeneous space $\mathscr{D}$ by some element $g \in SL_2\mathbb{R}$. By Floquet theorem we must take one and the same element g for all points $x \in \alpha$. In case the conditions of theorem 12.3 are true then $g \in SL_2\mathbb{R}$ is hyperbolic. Therefore the cascade $\{T_{n\tau}\}$ coincides with the cascade of the classical geodesic flow $\{R_t\}$ on the homogeneous space $\mathscr{D}$. It turns out that trajectories of a $\mathscr{N}g$-flow on ergodic components $\alpha \times \mathscr{D}$ are winding around the trajectories of the flow $Id \times R_t$ on $x \times \mathscr{D} \subset \alpha \times \mathscr{D}$.

Complete investigation of $\mathscr{N}g$-flow on $SL_2\mathbb{R}$ is given in [19]; in [17] description of $\mathscr{N}g$-flows for all 3-dimensional nonholonomic Lie groups and their compact homogeneous spaces is given.

15. NONHOLONOMIC EXPONENTIAL MAPPING. SINGULARITIES OF NONHOLONOMIC WAVE FRONTS

Assume (M, V) is a nonholonomic manifold. By a wave ϵ-front with point x as a center, we mean the set of end points of nonholonomic geodesics of length ϵ, starting in x. We denote this ϵ-front by $A_\epsilon^V(x)$. (Some authors use another name for $A_\epsilon^V(x)$ – geodesic ϵ-sphere). For Riemann manifolds wave ϵ-front coincides with ϵ-sphere for sufficiently small ϵ. In nonholonomic case these sets are different. We'll describe the wave front for the simplest case – three-dimensional nonholonomic Lie groups.

First define a *nonholonomic exponential mapping* $exp_\epsilon^x(v, \omega)$ as a mapping, transforming $t \in \mathbb{R}^1$ into the end point of the nonholonomic geodesic, starting from x with initial data $(v, \omega) \subset S_1 \times V^\perp_{(x)}$, where $S_1 \subset V(x)$ is a unit sphere. A wave front is equal to the image of the cylinder $S_1 \times V^\perp$ with respect to this nonholonomic exponent: $A_\epsilon^V(x) = exp_\epsilon^x(S_1 \times V^\perp)$.

For example, for a 3-dimensional Lie group we obtain that $A_\epsilon^V(e) = exp_\epsilon^e(S^1 \times \mathbb{R}^1)$. We xant to describe singularities of this exponential mapping exp_ϵ^e. Consider first a 3-dimensional Heisenberg group N_3. Let's realize this group as $\mathbb{R}^3$ (see 13).

PROPOSITION 15.1. [20]. (On wave front of Heisenberg group).

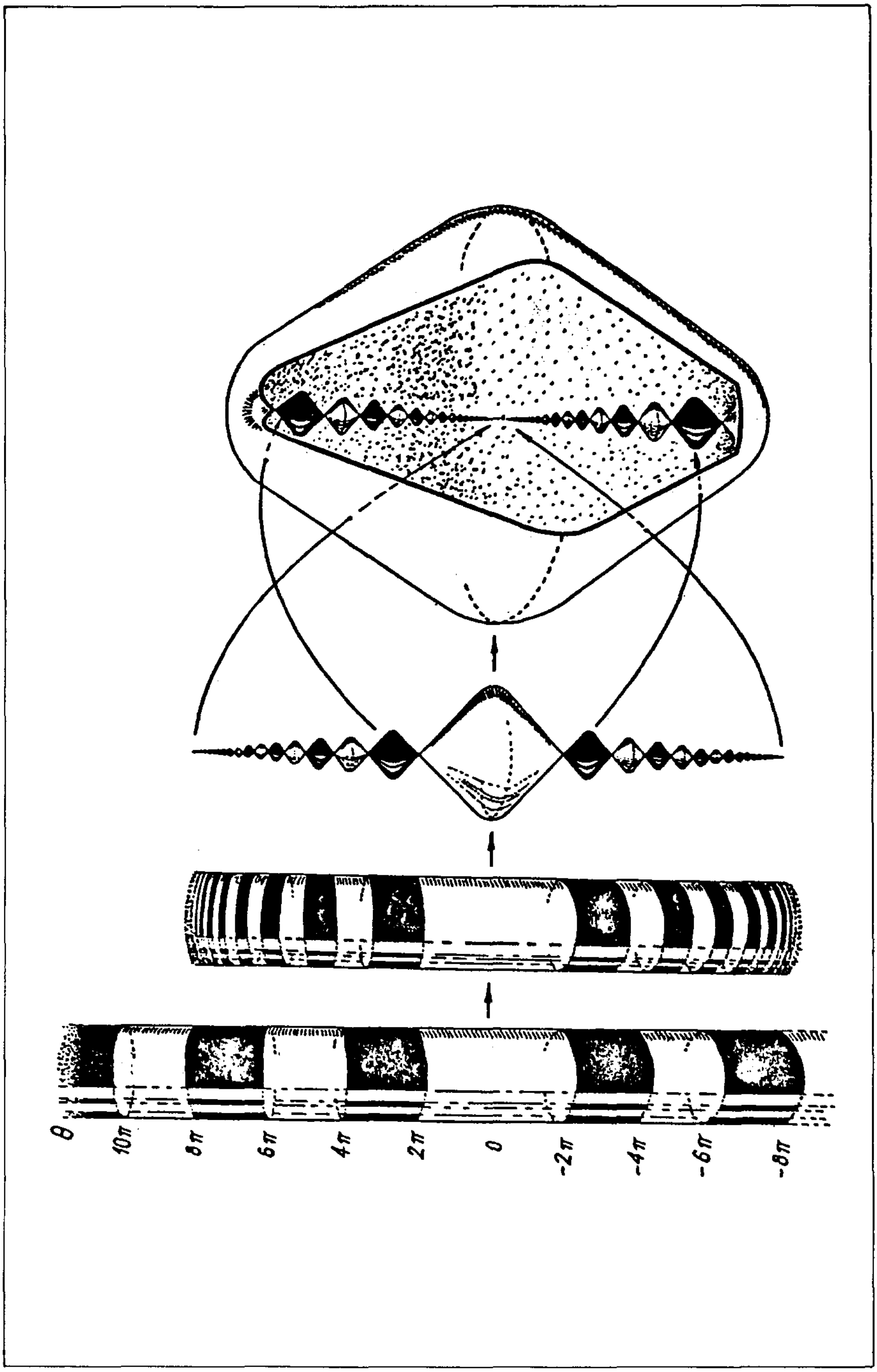

Fig. 1. Nonholonomic wave front for three dimensional Lie groups.

1. The circles C_n on $S^1 \times R^1 = \{(\varphi, \lambda)\}$, corresponding to $\lambda_n = 2\pi n/\epsilon$, $n = \pm 1, \pm 2, \ldots$, are transformed by the nonholonomic exponential mapping into the points $\Theta_n = (0, 0, \epsilon^2/4\pi n)$ belonging to the center of N_3.

2. In every point Θ_n the wave front $A_\epsilon^V(e)$ has a conic singularity (i.e. the germ of the wave front $A_\epsilon^V(e)$ in any point Θ_n, $n = \pm 1, \pm 2, \ldots$ is diffeomorphic to the germ of the circular cone in $0 \in \mathbb{R}^3$.

The nonholonomic exponential mapping of the cylinder $S^1 \times \mathbb{R}^1$ to $A_\epsilon^V(e)$ is schematically depicted in Fig. 1. The wave front $A_\epsilon^V(e)$ is a collection of beads, in the center of the Heisenberg group. These beads can be enumerated: by integers: n-th bead B_n is the image of the cylinder $S^1 \times [2\pi n, 2\pi(n+1)]$ if $n > 0$, and of $S^1 \times [2\pi(n-1), 2\pi n]$ if $n < 0$. A zeroth bead B_0 is the image of the cylinder $S^1 \times [-2\pi, 2\pi]$; B_0 coincides with the nonholonomic ϵ-sphere $S_\epsilon^V(e)$, i.e.

$$S_\epsilon^V(e) = exp_\epsilon^e\ (S^1 \times [-2\pi, 2\pi]).$$

From the formula for Θ_n one can see that all the beads lie inside B_0; the beads B_n condense themselves to the unit element of the group as $|n| \to \infty$(see Fig. 1) (compare with [41]).

THEOREM 15.3. (on singularities of wave fronts).

For all 3-dimensional nonholonomic Lie groups the wave front $A_\epsilon^V(e)$ is for small ϵ diffeomorphic to the wave front of Heisenberg group.

More detailed exposition, as well as the proof of all these statements, are given in [20]. Methods of this work allow to descript singularities of wave fronts for multidimensional nonholonomic Heisenberg groups with arbitrary left-invariant metric. These singularities are calculated by the same methods as for dim = 3, althoug multidimensional calculations are more technically complicated. For other groups with dim > 3 singularities are not yet calculated. Also nothing is known about singularities of wave fronts of nonholonomic Riemann manifolds (of course, except the case when they are locally isomorphic to nonholonomic groups). Even the case of contact structure in $\mathbb{R}^3$ with arbitrary metric is not yet investigated.

16. NONHOLONOMIC LAPLACIAN

With every Riemann metric on a smooth manifold M an elliptic operator is naturally associated – the Laplace-Beltrami operator Δ; vice versa, every elliptic operator of second order determines a Riemann metric on the manifold. We'll show that a likewise result is true for nonholonomic manifolds (M, V): namely, with every positive definite quadratic form on the distribution V a hypoelliptic operator Δ_V is naturally associated; it is natural to call this operator *Laplacian,* or *Laplace-Beltrami operator* of the *nonholonomic manifold.* We give three equivalent definitions of the operator Δ_V, that are likewise to three classical definition of Δ.

a) Assume M is a germ of a Riemannian manifold in some point $x \in M$, U-open neighbourhood of x, $\xi_1, \ldots, \xi_n$ – orthonormed frame of vector fields in U. The germ $\Delta|_U$ of the operator Δ is given by a formula

$$\Delta = -\sum_{i=1}^{n} \xi_i^2$$

For a germ of a nonholonomic Riemann manifold (M, V) it is natural to define the germ of Δ_V by a likewise formula

$$\Delta_V = \sum_{i=1}^{n_1} \xi_i^2$$

where $\{\xi_i\}_{i=1}^{n_1}$ is an orthogonal frame of vector fields for distribution V. Due to the fact that distribution V is completely nonholonomic, the conditions of Hermander's theorem on sum of squares are fulfilled (see [37]) and we obtain the following statement.

THEOREM 16.1. Assume (M, V) is a nonholonomic Riemann manifold. Then Δ_V is a hypoelliptic operator.

b) Laplace-Beltrami operator can be also defined by a formula

$$-\Delta U = \mathit{div}\, \mathit{grad}\ U.$$

In order to transfer this formula to a nonholonomic case denote $\mathit{grad}_V = P_V\, \mathit{grad}$ where $P_V : TM \to V$ is the orthogonal projector on V, and $\mathit{div}_V = \mathit{div} \cdot P_V$.

PROPOSITION 16.2. Assume (M, V) is a nonholonomic manifold. Then $\Delta_V = \mathit{div}_V\, \mathit{grad}_V$.

This statement shows that Δ_V does not depend on the concrete choice of the orthogonal frame; this propositions allows to define Δ_V for the whole manifold.

c) The operator Δ can be also defined by means of a differential complex: $\Delta = d\delta + \delta d$ (see e.g. [38]). Denote $d_V = P_V d$ and $\delta_V = \delta P_V$.

PROPOSITION 16.3.

$$\Delta_V = d_V \delta_V + \delta_V d_V$$

Remarks 1. These definitions of Δ_V can be applied also in case when the distribution V is not completely nonholonomic. For example, if V is integrable, it determine a foliation $\mathscr{V}$ on the manifold M then Δ_V determines a family of Laplacians on fibres of $\mathscr{V}$.

2. Assume $V^\perp$ is an orthogonal completion to V in TM. Then $\Delta = \Delta_V + \Delta_{V^\perp}$.

We'll say that two distributions V, W commute, if there exist such orthogonal lases $\{\xi_i\}$ and $\{\psi_j\}$ correspondingly for distributions V and W that $[\xi_i, \psi_j] = 0$ for all i and j.

PROPOSITION 16.4. If distributions V and $V^\perp$ on a manifold M commute, that the operators $\Delta_V, \Delta_{V^\perp}, \Delta$ commute with each other.

In this case Laplacian Δ_V has the same set of eigenfunctions as operator Δ.

Example. Assume G is a 2-step nilpotent Lie group, V – canonical distribution on G. Then Δ_V and Δ commute. In particular, Laplacian Δ_V of a contact structure commutes with Δ.

17. HOPF-ALEXANDROV THEOREM

Classical Hopf theorem (see e.g. [34]) states that on closed manifolds there are no harmonic functions (i.e. solutions of the equation $\Delta U = 0$, that are different from constants.

Let's call a function U on a nonholonomic manifold (M, V) *V-harmonic*, or *hypoharmonic*, if $\Delta_V U = 0$ A.D. Alexandrov proved the following generalization of Hopf theorem to the nonholonomic case.

THEOREM 17.1. (On hypoharmonic functions). Assume (M, V) is a closed nonholonomic Riemann manifold. Then every hypoharmonic function on M is constant.

Let's give a short proof of this theorem, that follow the classical example and at the same time demonstrates the use of nonholonomic technique. Calculate

$$\Delta_V(f^2) = div\, P_V\, grad\, f^2 = div\, P_V\, 2f\, grad\, f =$$
$$= div\, 2f\, grad_V\, f = \langle grad_V\, f, grad_V\, f\rangle + 2f \Delta_V f$$

In case f is hypoharmonic, we obtain

$$\langle grad_V\, f, grad_V\, f\rangle = div\, grad_V\, f^2$$

Due to Green theorem (see e.g. [34])

$$0 = \int_M div\, grad_V f^2\, dm = \int_M \langle grad_V f, grad_V f\rangle\, dm$$

(here dm is a Riemann measure on M). Therefore $P_V grad f = 0$, i.e. $grad f$ is orthogonal to the distribution V therefore f remains constant along arbitrary admissible curve in M. According to Rashevsky-Chow theorem (see 7) every pair of points from M can be connected by an admissible curve. Therefore f is constant on M.

Remark. This theorem can be easily modified to the case when the distribution V is not completely nonholonomic (see [3, 22]).

Another proof of this theorem can be obtained from Alexandrov's maximum principle for hypoharmonic functions.

THEOREM 17.2. (Alexandrov's maximum principle). Assume a hypoharmonic function is defined on some open domain U. Then it cannot have maximum inside U.

Summarizing: The subspace of hypoharmonic functions is onedimensional and includes only constants.

18. NONHOLONOMIC GREEN FORMULA

Assume (M, V) is a nonholonomic Riemann manifold, dm – Riemann measure on M.

Then the following analogue of Green formula is true.

PROPOSITION 18.1. Assume (M, V) is a nonholonomic Riemann manifold. Then the following formulas are true:

1) operator Δ_V is selfadjoint in $L^2(M)$, i.e.

$$\int_M \omega\, \Delta_V u\, dm = \int_M u\, \Delta_V\, \omega\, dm,$$

2)
$$\int_M u\, \Delta_V\, \omega\, dm = \int_M \langle grad_V\, u,\ grad_V\, \omega\rangle\, dm$$

Formulas 1.2 show that Δ_V is a positive definite selfadjoint operator.

PROPOSITION 18.2. (Metivier [2]). Assume (M, V) is a nonholonomic compact Riemann manifold. Then the operator Δ_V has a positive discrete spectrum.

Denote the elements of this spectrum by $\lambda_1^V \leqslant \lambda_2^V \leqslant \ldots$ It turns out that the asymptotics of this spectrum depends only on its principal quasihomogeneous part and, therefore, the problem of finding this asymptotics can be reduced to the same problem for nilpotent Lie groups (for details see 19).

19. PRINCIPAL QUASIHOMOGENEOUS PART OF THE NONHOLONOMIC LAPLACIAN

Some preliminary definitions are necessary.

Assume (M, V) is a germ of a nonholonomic manifold in some point x, N-growth vector of the distribution V. The filtration, induced by the vector N on the Lie algebra J_x *Vect* of jets of vector fields (see 5, 6) can be naturally extended to its universal envelopping algebra $\mathcal{U}_N$. The corresponding graduation allows to represent $\mathcal{U}_N$ as a direct sum of quasihomogeneous component

$$\mathcal{U}_N = \bigoplus_{j=-\infty}^{\infty} \mathcal{U}^j$$

where $\mathcal{U}^j$ is generated by monomials

$$x_1^{l_1} \cdot \ldots \cdot x_n^{l_n} \; \frac{\partial^{m_1}}{\partial x_1^{m_1}} \cdot \ldots \cdot \frac{\partial^{m_n}}{\partial x^{m_n}}$$

that satisfy the condition

$$\sum_{i=1}^{n} (l_i - m_i)\, \varphi_V(i) = j$$

By ρ_i we denote the projector $\rho_i : \mathcal{U}_N = \oplus\, \mathcal{U}^j$ on i-th component of this sum. The above-mentioned filtration consists of $\mathcal{U}_j = \bigoplus_{i \geqslant j} \mathcal{U}^i$. The action of the group of quasihomogeneous dilatations can be naturally extended to $\mathcal{U}_N$.

In these terms nonholonomic Laplacian

$$\Delta_V = \sum_{i=1}^{n_1} \xi_i^2$$

belongs to $\mathcal{U}_{-2} \subset \mathcal{U}_N$. By a principal quasihomogeneous part of the Laplacian we'll understand the operator $p_{-2}\, \Delta_V \in \mathcal{U}^{-2}$. Let's denote it by $\Delta_V^{(0)}$.

PROPOSITION 19.1. Assume (M, V) is a germ of a nonholonomic manifold in some point x, $V^{(0)}$ – a germ (in point x) of a distribution, that is a principal quasihomogeneous part of a distribution V. Then $\Delta_V^{(0)} = \Delta_{V^{(0)}}$.

So the principal quasihomogeneous part of Laplacian in point x of a nonholonomic manifold can be naturally interpreted as a nonholonomic Laplacian on the germ of an osculating Lie group $(G_x, V_x) \simeq (M, V^{(0)})$.

So it is necessary to investigate properties of nonholonomic Laplacians on homogeneous nilpotent Lie groups G. The action $\{H_t\}$ of the quasihomogeneous dilatations group on Lie group G induces its action on the space $\mathcal{L}^2(G)$ of measurable functions on G with integrable square. Namely, for $f \in \mathcal{L}^2(G)$ we assume that

$$(\tilde{H}_t f)x = f(H_t x).$$

We denote quasihomogeneous dilatations on $\mathcal{L}^2(G)$ by $\tilde{H}_t$ in order to distinguish them from dilatations of differential operators; for which we preserve the denotation H_t.

PROPOSITION 19.2. Assume (G, V) is a nilpotent nonholonomic Lie group, and a left-invariant metric is given. Then the following formulas are true:

1) $\tilde{H}_{t^{-1}} \Delta_V \tilde{H}_t = t^2 \Delta_V$

2) $H_t \Delta_V = t^{-2} \Delta_V$

(i.e. $((H_t \Delta_V) f) x = t^{-2} (\Delta_V f) x$).

Let's now turn to spectrs and spectral functions. Remind (see e.g. [35]) that the spectral function $e(x, y, \lambda)$ of a self-adjoint operator A is a Kernel of its spectral projector $A_{(-\infty, \lambda)}$. In case of a discrete spectrum $e(x, y, \lambda)$ can be expressed as follows. Assume $S_p \Delta_V = \{\lambda_1^V \leqslant \lambda_2^V \leqslant \ldots\}$ is an ordered set of eigenvalues of the operator Δ_V, and $\{f_j\}_{j=1}^{\infty}$ is an orthonormed set of correspondent eigenfunctions.

$$e(x, y, \lambda) = \sum_{\lambda_j \leqslant \lambda} f_j(x) f_j(y)$$

(In case we consider several operators, we'll point out whose spectral function in considered by explicitly indicating this operator e.g. $e_{\Delta V}(x, y, \lambda)$ or $e_{\Delta}(x, y, \lambda)$.

PROPOSITION 19.3. Assume (G, V) is a nilpotent nonholonomic Lie group, Δ_V-Laplacian on (G, V); then the spectral function $e_{\Delta_V}(x, y, \lambda)$ satisfies the following conditions:

1) $e_{\Delta_V}(x, x, \lambda)$ does not depend on x

2) $e_{\Delta_V}(e, e, n^2\lambda) = e_{H_n^{-1}\Delta_V H_n}(e, e, \lambda)$

The first property is fulfilled for all left invariant operators on a Lie group, because then left shift of an eigenfunction by an element of the group is also an eigenfunction, with the same eigenvalue.

The next result of Metiver reduces the calculation of asymptotics of a spectral function $e(x, x, \lambda)$ for a nonholonomic Riemann manifold (M, V) to the group case.

THEOREM 19.3. [2]. Assume (M, V) is a germ of a nonholonomic Riemann manifold in some point x, $V^{(0)}$-the principal part of V in the point x. Then

$$e_{\Delta_V}(x, x, \lambda) \underset{\lambda \to \infty}{\sim} e_{V^{(0)}}(x, x, \lambda).$$

I.e. asymptotics of $e_{\Delta_V}(x, x, \lambda)$ is the same as for the spectral function of the Laplacian of the osculating Lie group.

In order to find this asymptotics we make the following definition. By $N_A(\lambda)$ we denote the total number of eigenvalues of operator A that do not exceed λ. The function N can be expressed in terms of a spectral function on a compact manifold by a formula.

$$N_A(\lambda) = \int_M e_A(x, x, \lambda)\, dm$$

Quasihomoneneous character of the spectral functions allows to show that for compact homogeneous space of a nilpotent nonholonomic Lie group (G, V) the function N_{Δ_V} has the following asymptotics:

$$N_{\Delta_V}(\lambda) \underset{\lambda \to +\infty}{\sim} p_G\, Vol_V\, G \cdot \lambda^{d_V/2}$$

where d_V is a homogeneous (Hausdorff) dimension of G, $Vol_V G$ its Haar volume and the constant p_G (we'll call it density) depend on the group G. (We'll investigate this characteristic in the next point). In these terms Metivier's theorem can be reformulated as follows.

THEOREM 19.4. Assume (M, V) is a germ of a nonholonomic Riemann manifold in some point x, Δ_V is a nonholonomic Laplacian on (M, V) and $e_{\Delta_V}(x, x, \lambda)$ its spectral function. Then the following limit exists

$$\lim_{\lambda \to \infty} \lambda^{-d_V/2} e(x, x, \lambda) = P_{G_x}$$

where d_V is a Hausdorff dimension of the nonholonomic manifold (M, V), G_x is a osculating nilpotent Lie group.

It is therefore natural to expect that the asymptotic formula for $N_{\Delta_V}(\lambda)$ can be generalized to the case of nonholonomic Riemann manifolds; in this generalization Haar measure on a group G must be changed to a measure on a nonholonomic Riemann manifold constructed by means of densities p_{G_x} (see 20) definding on osculating Lie groups, where Haar measure on G_x is induced by Riemann measure on M.

20. HYPOTHESIS ON MAIN PRINCIPAL IN WEYL FORMULA FOR NONHOLONOMIC LAPLACIAN

For Laplacian on the Riemann compact manifold M the asymptotics of eigenvalues growth is given by the classical Weyl formula (see for example [35]):

$$N_\Delta(\lambda) = \sum_{\lambda_i \leq \pi} 1 \sim C_d \ Vol\, M \cdot \lambda^{d/2} \ (1 + o\,(1))$$

where $d = dim\, M$, is the dimension of M, $Vol\, M$ is Riemann volume of M and c_d is the constant depending only on dimension of M. Metivier's theorem allows to present the principal term of asymptotics for the spectrum of the nonholonomic Laplacian on a compact nonholonomic manifold (M, V) in the following form:

$$N_{\Delta_V}(\lambda) = C_{(M,\, V)} \cdot \lambda^{d_V/2}(1 + o\,(1)) \tag{20.1}$$

where d_V is the Hausdorff dimension of the nonholonomic manifold (M, V) and

$$C_{(M,\, V)} = \int_M P_{G_x} d_m$$

where d_m is the Riemann measure on M, p_{G_x} is the density on osculating Lie group G_x (see 19). So

$$N_{\Delta_V}(\lambda) = \left(\int_M p_{G_x} d_m \right) \lambda^{d_V/2} \ (1 + o\,(1)) \tag{20.2}$$

It is natural to define a nonholonomic volume of a compact nonholonomic manifold (M, V) as

$$\int_M p_{G_x} d_m .$$

THEOREM 20.1. Let (M, V) be a nonholonomic Riemann manifold, Δ_V be the nonholonomic Laplacian on (M, V). Then the spectrum of Δ_V has the following asymptotics

$$N_{\Delta_V}(\lambda) \underset{\lambda \to \infty}{\sim} Vol_V M \cdot \lambda^{d_V/2}$$

Let's turn to the interpretation of the density p_{G_x}. Let (G, V) be a nonholonomic nilpotent Lie group. Δ_V – the nonholonomic Laplacian on G and $\xi(t)$ be the diffusion Markov process on G with initial state $\xi(0) = e$ and generator Δ_V. Denote by $p_t(x, y)$ probability density of the transition from x to y during time t. This density is calculated with respect to Haar measure on the osculating Lie group G_x. In particular $p_t(e, e)$ means the probability density of returning to the unit $e \in G$ during time t.

PROPOSITION 20.2. For small t the density $p_t(e, e)$ has the following asymptotics:

$$p_t(e, e) = p_{G_x} \cdot t^{-d_V/2}\,(1 + o(1))$$

where p_{G_x} – density for Lie group G_X.

The proof of proposition is standard. So the problem about principal term of generalized Weyl's formula is reduced to (nilpotent) group case. For the simplest nilpotent groups this problem was solved by Gaveau [5]. We shall return to the question elsewhere (*).

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(*) Explicit calculations of eigenvalues of nonholonomic Laplacian on Heisenberg group see in [39].

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